Linear, affine, conic and convex hulls

Let $X \subseteq \mathbb{R}^n$:

$$
\begin{aligned}
\mathrm{lin.\,hull}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \in \mathbb{N}_0, \\
&\qquad x_1, \ldots, x_t \in X,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}\} \\
\mathrm{affine.\,hull}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \in \mathbb{N}_+, \\
&\qquad x_1, \ldots, x_t \in X,\ \sum_{i=1}^{t} \lambda_i = 1,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}\} \\
\mathrm{cone}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \in \mathbb{N}_0, \\
&\qquad x_1, \ldots, x_t \in X,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}_{\geqslant 0}\} \\
\mathrm{conv}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \in \mathbb{N}_+, \\
&\qquad x_1, \ldots, x_t \in X,\ \sum_{i=1}^{t} \lambda_i = 1,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}_{\geqslant 0}\}
\end{aligned}
$$

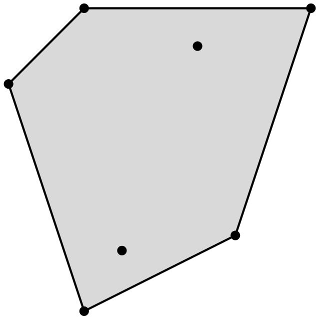

Figure: The convex hull of 7 points in $\mathbb{R}^2$.

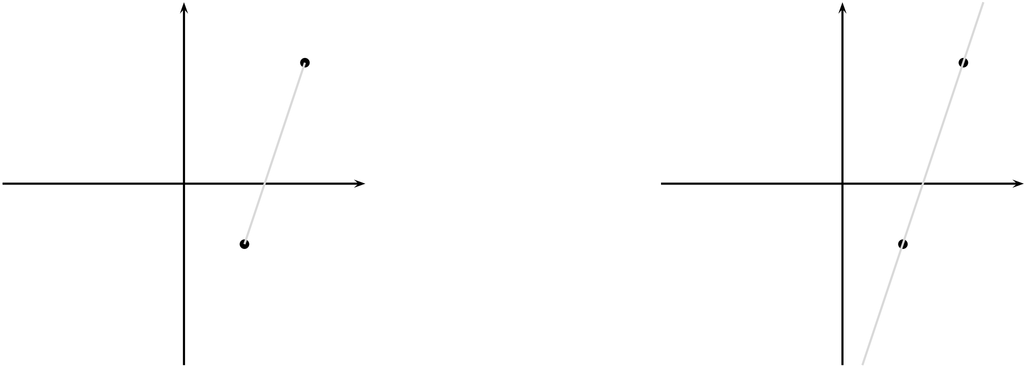

Figure: Two points with their convex hull on the left and their affine hull on the right.

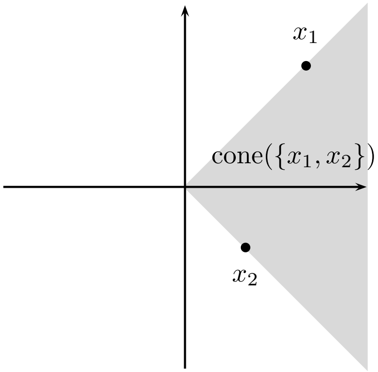

Figure: Two points with their conic hull

Linear and affine hulls

Theorem

Let $X \subseteq \mathbb{R}^n$ *and* $x_0 \in X$. *One has*

$$\operatorname{affine.hull}(X) = x_0 + \operatorname{lin.hull}(X - x_0),$$

where for $u \in \mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$, $u + V$ denotes the set $u + V = \{u + v \mid v \in V\}$.

Convex hull is convex

Theorem

Let $X \subseteq \mathbb{R}^n$ be a set of points. The convex hull, $\mathrm{conv}(X)$, of X is convex.

Convex hull is minimal

Theorem

Let $X \subseteq \mathbb{R}^n$ be a set of points. Each convex set K containing X also contains $\mathrm{conv}(X)$.

Corollary

$$\operatorname{conv}(X) = \bigcap_{\substack{K \supseteq X \\ K \text{ convex}}} K .$$

Cones

Definition

A set $C \subseteq \mathbb{R}^n$ is a cone, if it is convex and for each $c \in C$ and each $\lambda \in \mathbb{R}_{\geqslant 0}$ one has $\lambda \cdot c \in C$.

Analogous theorems for cones

Theorem

For any $X \subseteq \mathbb{R}^n$, the set $\mathrm{cone}(X)$ is a cone.

Theorem

Let $X \subseteq \mathbb{R}^n$ be a set of points. Each cone containing X also contains $\mathrm{cone}(X)$.

$$\mathrm{cone}(X) = \bigcap_{\substack{C \supseteq X \\ C \text{ is a cone}}} C.$$

Carathéodory's Theorem

Theorem

Let $X \subseteq \mathbb{R}^n$, then for each $x \in \operatorname{cone}(X)$ there exists a set $\widetilde{X} \subseteq X$ of cardinality at most n such that $x \in \operatorname{cone}(\widetilde{X})$. The vectors in $\widetilde{X}$ are linearly independent.

Bounded continuous functions

Theorem

Let $X \subseteq \mathbb{R}^n$ be compact and $f : X \to \mathbb{R}$ be continuous. Then f is bounded and there exist points $x_1, x_2 \in X$ with $f(x_1) = \sup\{f(x) \colon x \in X\}$ and $f(x_2) = \inf\{f(x) \colon x \in X\}$.

Separation theorem

Theorem

Let $K \subseteq \mathbb{R}^n$ be a closed convex set and $x^* \in \mathbb{R}^n \setminus K$, then there exists an inequality $a^T x \leq \beta$ such that $a^T y < \beta$ holds for all $y \in K$ and $a^T x^* > \beta$.

Farkas' Lemma – Version 1

Theorem (Farkas' lemma)

Let $A \in \mathbb{R}^{m \times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax = b$, $x \geqslant 0$ has a solution if and only if for all $\lambda \in \mathbb{R}^m$ with $\lambda^T A \geqslant 0$ one has $\lambda^T b \geqslant 0$.

Farkas' Lemma – Version 2

Theorem (Farkas' lemma)

Let $A \in \mathbb{R}^{m \times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax \leq b$ has a solution if and only if for all $\lambda \in \mathbb{R}^m_{\geqslant 0}$ with $\lambda^T A = 0$ one has $\lambda^T b \geqslant 0$.

Exercise!