Königs Thm:

Def: Let $G=(V,E)$ graph. $S \subseteq V$ is vertex cover if

$\forall e \in E \quad e \cap S \neq \emptyset$

Example:

Thrm: (Königs theorem) Let $G=(A+B, E)$ be a bipartite graph.

The max-cardinality of a matching in G = min-card. of vertex cover.

proof:

$$\max \sum_{e \in E} x(e) \qquad\qquad \min \sum_{v \in V} y(v)$$

$$v \in V: \quad \sum_{e \in \delta(v)} x_e \leq 1 \qquad\qquad uv \in E: \quad y(u) + y(v) \leq 1$$

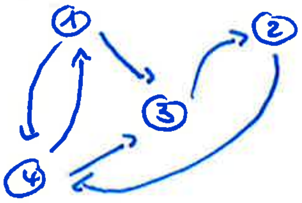
$$x \geq 0 \qquad\qquad y \geq 0$$

Both TU opt. 0/1 sol. of LPs exist. ▨

Network Flow

Def: $D = (V, A)$ is directed graph if V is finite set of nodes and $A \subseteq V \times V$ is set of arcs.

Example:



$V = \{1, 2, 3, 4\}$

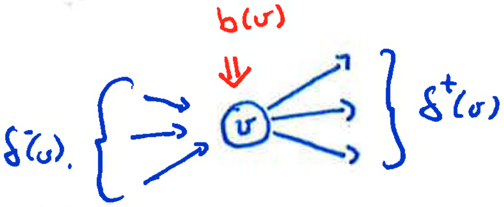
$A = \{(1,4), (4,1), (1,3), (4,3), (3,2), (2,4)\}.$

Definition: Network Flow is a vector $f \in \mathbb{R}^A$ s.th.

$$\forall v \in V \qquad \sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e) = b(v) \qquad (*)$$

$$f \geq 0$$

where $b \in \mathbb{R}^{|V|}$ is given.



Definition: Min-Cost Network Flow Problem

Given: $D = (V, A)$, $b \in \mathbb{R}^{|V|}$, $u \in \mathbb{R}^{|A|}_{\geq 0}$, $c \in \mathbb{R}^{|A|}$

$$\min \; c^T \cdot f$$

f sat. $(*)$ and $f \leq u$

capacities.

Necessary condition on existence of feasible flows:

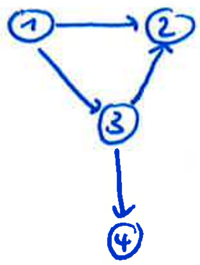
$$\sum_{v \in V} b(v) = 0$$

summing up $(*)$ each arc $e = (u) \to (v)$ appears twice once positive (for u) once negative (for (v))

Def: $A_G \in \{0, \pm 1\}^{|V| \times |A|}$ is directed node-edge incidence matrix, where

$$A_G(u,e) = \begin{cases} 1 & \text{if } e = (u,x) \\ -1 & \text{if } e = (x,u) \\ 0 & \text{otherwise.} \end{cases} \quad \text{for some } x \in V$$

Example



$$A_G = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Min-Cost Network Flow problem:

$$\min c^T \cdot f$$
$$A_G \cdot f = b$$
$$0 \leq f \leq u.$$

Theorem: A_G is totally unimodular.

To show: Each $k \times k$ sub-matrix A' of A satisfies $\det(A') = \pm 1, 0$

Proof: induction: A' 1×1 submatrix $\Rightarrow \det(A') = \pm 1, 0$

CASE 1: A' contains column with exactly one entry $\neq 0$

$$A' = \begin{pmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ & \pm 1 & \\ & 0 & \\ & \vdots & \\ & 0 & \end{pmatrix} \Rightarrow \det(A') = (\pm 1) * \det(A'')$$

develop.

A'' $(k-1) \times (k-1)$ sub-matrix.

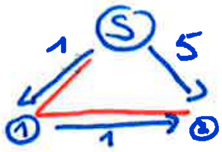
induction $\det(A'') = \pm 1, 0.$

Example:

Given: $G=(V,A)$, $c: A \to \mathbb{R}^+$, $s \in V$

Problem: Compute shortest path distances from s to all other nodes $u \in V \setminus \{s\}$.

example:

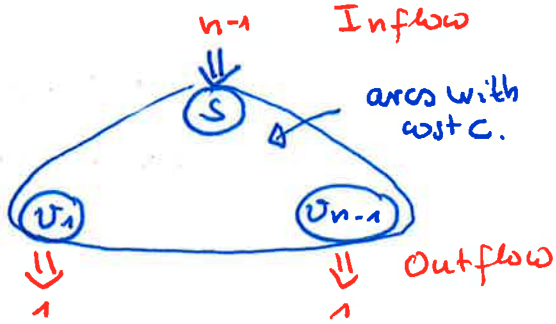


$d(s,1)=1$
$d(s,2)=2$

Model as min-cost Network flow:

$$\min \; c^T \cdot f$$

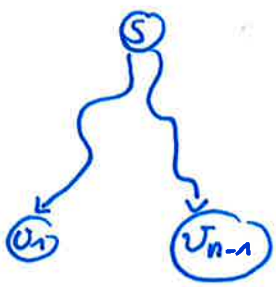
f feasible flow in network.



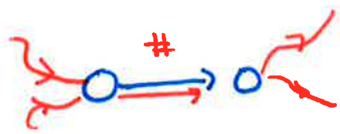
$$\boxed{\begin{aligned} &\sum_{e \in \delta^+(s)} f(e) - \sum_{e \in \delta^-(s)} f(e) = n-1 \\ \forall v \in V\setminus\{s\} \quad &\sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e) = -1 \\ &0 \le f \end{aligned}} \quad (*)$$

Lemma: $(*)$ feasible $\Leftrightarrow$ $\exists$ s,v-Path in G $\forall v \in V$.

proof: "$\Leftarrow$"



n-1 paths:

$e=$ 

$f(e) = \#$ of Paths using e.

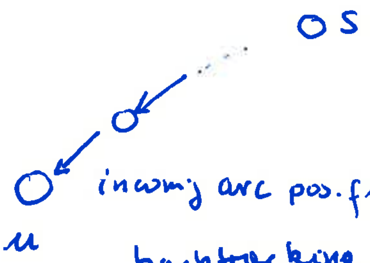
f feasible flow.

„=>“ let f be feasible flow and optimal flow.

=> f has no cycle.

$f' = f - \delta \cdot$ Cycle ~~of~~

still feasible and of strictly smaller cost.



backtracking ends in s, since no cycle in f.

□

<u>Theorem:</u> ~~Ex~~ optimal solution is of the form

$$f_{opt} = x(P_{v_1}) + \dots + x(P_{v_{n-1}})$$

where ~~to~~ P_{v_i} is shortest path from s to v_i.

<u>proof:</u> Backtracking:

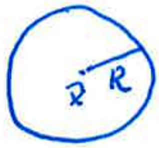
incoming pos. flow integral ≥ 1.

Ships $x(P_u)$.

∎

The ellipsoid method

Def: $B(\hat{x}, R) = \{ x \in \mathbb{R}^n : \|x - \hat{x}\| \leq R \}$



Ellipsoid Method : What it does.

Given :

i) $R \in \mathbb{R}_+$, ii) $K \subseteq B(0, R)$ convex with volume iii) $\mu \in \mathbb{R}_+$

find $x^* \in K$ or assert $vol(K) < \mu$

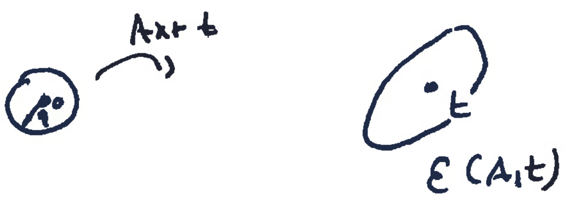
Ellipsoid method proceeds in iterations.

Number of iterations: $\boxed{O\left(\log \frac{R}{\mu} \cdot poly(n)\right)}$

Definition: Let $A \in \mathbb{R}^{n \times n}$ non-singular and $t \in \mathbb{R}^n$.

$\mathcal{E}(A,t) \subseteq \mathbb{R}^n$ ~~is~~ image of $B(0,1)$ under map $f: \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax + t$

$\mathcal{E}(A,t)$ is an ellipsoid



<u>Exercise:</u> $\mathcal{E}(A,b) = \{ x \in \mathbb{R}^n \mid \| A^{-1} \cdot x - A^{-1} \cdot t \|_2 \leq 1 \}$.

$$vol(B(0,1)) \sim \frac{1}{\pi n} \left(\frac{2e\pi}{n} \right)^{n/2} := V_n .$$

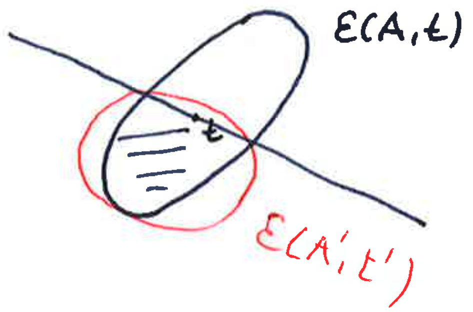
$$vol(\mathcal{E}(A,b)) = |det(A)| \cdot V_n .$$

<u>Theorem:</u> Let $\mathcal{E}(A,t)$ be an ellipsoid and $a \in \mathbb{R}^n$ WoS.

The half-ellipsoid $\mathcal{E}(A,t) \cap \{ x \in \mathbb{R}^n : a^T \cdot x \leq a^T \cdot t \}$

is contained in an ellipsoid $\mathcal{E}(A',t')$ s.t.

$$\frac{vol\ \mathcal{E}(A',t')}{vol(\mathcal{E}(A,t))} \leq \frac{1}{e^{\frac{1}{2(n+1)}}}$$



<u>proof</u> Later!

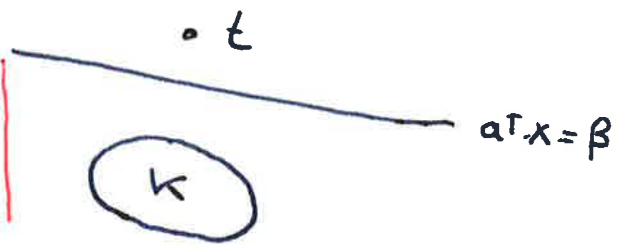
The method:

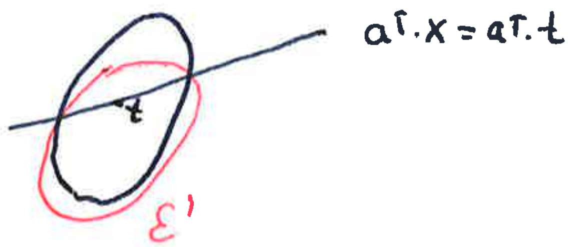
- $\mathcal{E} := B(0,R)$
- while $vol(\mathcal{E}) \geq \mu$
  - Let $\mathcal{E} = \mathcal{E}(A,t)$.
  - if $t \in K$: **return t**

    else : find $a \in \mathbb{R}^n \setminus \{0\}$, $\beta \in \mathbb{R}$

    s.th.

    $\boxed{a^T \cdot x = \beta \text{ separates } t \text{ from } K.}$





  - Set $\mathcal{E} := \mathcal{E}'$ from *half-ellipsoid lemma*.

Invariant: $K \subseteq \mathcal{E}$

Running time: After i iterations:

$$\mu \leq vol(\mathcal{E}) \leq \underline{V_n \cdot R^n} \cdot \frac{1}{e^{\frac{i}{2(n+1)}}}$$

$$\Leftrightarrow \quad e^{\frac{i}{2(n+1)}} \leq \frac{V_n \cdot R^n}{\mu}$$

$$\Leftrightarrow \quad \frac{i}{2(n+1)} \leq \frac{\ln(V_n) + n \cdot \ln(R)}{\ln(\mu)} \quad \Leftrightarrow \quad i \leq 2(n+1) \frac{\ln(V_n) + n \cdot \ln(R)}{\ln(\mu)}$$

The number of iterations is bounded by

$$\mathrm{poly}(n) \cdot \frac{\ln(R)}{\ln(\mu)}$$