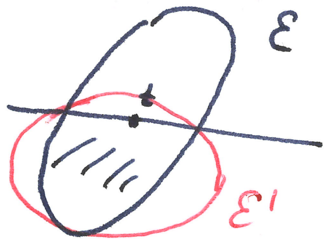
Half ellipoid theorem:



$$\frac{vol(\mathcal{E}')}{vol(\mathcal{E})} \leq \left(\frac{1}{e}\right)^{\frac{1}{2(n+1)}}$$

Half-ball lemma:

$H = \{x \in \mathbb{R}^n : \|x\| \leq 1, x_1 \geq 0\}$ contained in

$$\mathcal{E} = \{x \in \mathbb{R}^n : \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2 \leq 1\}$$

$$vol(\mathcal{E}) = |det(A)| \cdot vol(B(0,1))$$

with $A = \begin{bmatrix} \frac{n}{n+1} & & & \\ & \sqrt{\frac{n^2}{n^2-1}} & & \\ & & \ddots & \\ & & & \sqrt{\frac{n^2}{n^2-1}} \end{bmatrix}$

$$|det(A)| \leq \frac{1}{e}^{\frac{1}{2(n+1)}}$$

<u>proof:</u> To show: $x \in \mathbb{R}^n$, $\|x\| \leq 1$ and $x \geq 0$

$$\Rightarrow \quad \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \dots + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2 \leq 1.$$

One has $\displaystyle\sum_{i=2}^{n} x_i^2 \leq 1 - x_1^2$

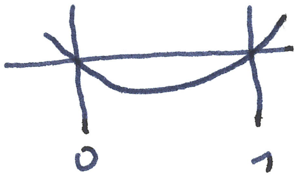
Define $f(x) = \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}\left(1 - x_1^2\right)$

$$f(0) = \frac{1}{n^2} + \frac{n^2-1}{n^2} = 1$$

$$f(1) = \left(\frac{n+1}{n}\right)^2 \left(\frac{n}{n+1}\right)^2 + 0 = 1$$

$f(x)$ is degree two polynomial. Thus assertion follows from

$f'(0) \leq 0$



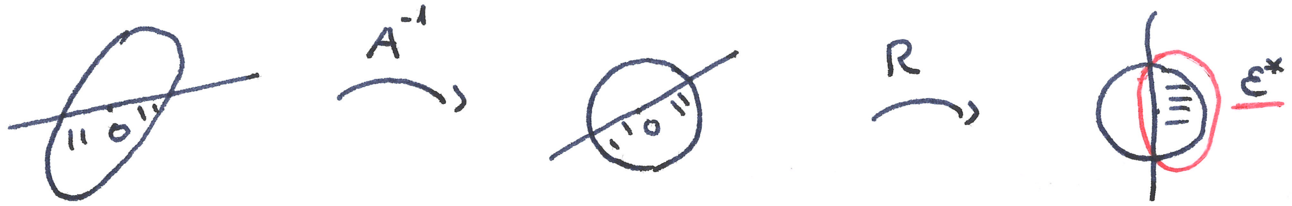
[$f'(x)$ only one root.]

$$f'(0) = 2 \cdot \left(\frac{n+1}{n}\right)^2 \left(0 - \frac{1}{n+1}\right) < 0$$

∎

Half-ellipsoid theorem: Sketch of proof.

$\mathcal{E} = \mathcal{E}(A, 0)$



$\mathcal{E}' = A \circ R^{-1}(\mathcal{E}^*) \longrightarrow$

Detailed Formulae as exercise.

WARNING: Details are missing in our treatment.

i) How to ensure that numbers do not grow too much?

ii) How to avoid $\sqrt{\cdot}$ computations?

Theorem: Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ there is a polynomial-time algorithm that either

- finds $x^* \in \mathbb{R}^n$ $A \cdot x^* \le b$
- OR ASSERTS THAT $P = \{Ax \le b\}$ is not full-dimensional.

Theorem: There is a polynomial time (in $\log(B)+n$) algorithm that decides whether $\{Ax \leq b\} \neq \emptyset$.

proof:

Recall Farkas: $\{Ax \leq b\} = \emptyset \iff$

$$\left.\begin{aligned} \lambda^T \cdot b &= -1 \\ \lambda^T \cdot A &= 0 \\ \lambda &\geq 0 \end{aligned}\right\} \text{ feasible.}$$

In this case, there exists λ basic feasible solution.

$\Rightarrow \quad \exists \quad \lambda \in \mathbb{Z}_{\geq 0}$

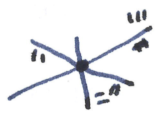
$$\begin{aligned} \lambda^T \cdot b &\leq -1 \\ \lambda^T \cdot A &= 0 \\ \lambda &\geq 0 \end{aligned} \qquad \|\lambda\|_\infty \leq (n \cdot B)^n$$

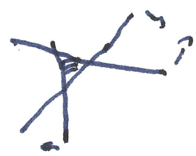
Auxiliary system: $\quad Ax \leq b + \varepsilon \cdot \mathbb{1} \qquad$ (I)

$$\varepsilon = \frac{1}{2} \cdot \frac{1}{n \cdot (n \cdot B)^n}$$

(I) feasible $\iff$ $\{Ax \leq b\}$ feasible.

if (I) feasible $\Rightarrow$ (I) full-dimensional



$\rightarrow$ 

Run Ellipsoid method on (I)

∎

Thm: There exists a polynomial time (in $\log(B)+n$)

algorithm that, given $Ax \leq b$ $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$

- finds $x^* \in \mathbb{R}^n$: $Ax^* \leq b$
- asserts $\{Ax \leq b\} = \emptyset$.

proof: Run alg. ~~on~~ to decide whether $\{Ax \leq b\} \neq \emptyset$

If $\neq \emptyset$, then search for the basis (w.L.O.G. A full

col.-rank).

try $a^T x \leq \beta$ of $Ax \leq b$

Test $Ax \leq b$, $a^T x = \beta$ for feasibility, if yes, keep

$a^T x = \beta$, otherwise delete $a^T x \leq \beta$ from $Ax \leq b$

try next ineq $a' x \leq \beta'$:

$Ax \leq \beta$, $a^T x = \beta$, $a' x = \beta'$

$\vdots$

$\blacksquare$

Theorem: Linear programming can be solved in polynomial time.

proof:

$$\max c^T \cdot x$$
$$Ax \leq b$$

Solve feasibility problem of

$$\boxed{\begin{array}{ll} c^T \cdot x = b^T \cdot y & \\ Ax \leq b, & A^T \cdot y = c \\ & y \geq 0 \end{array}}$$

Primal and dual coupled.

$\blacksquare$