Recap: Integer Program

$$\max \; c^T \cdot x$$
$$Ax \le b$$
$$x \in \mathbb{Z}^n \quad \leftarrow \text{ linear Program: } \mathbb{R}^n$$

Allows to model decisions.

Example: Max weight matching. $G = (V, E)$, $w: E \to \mathbb{R}$

task:
$$\max_{\substack{M \subseteq E \\ M \text{ matching}}} \sum_{e \in M} w(e).$$

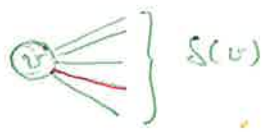
Variables $x(e) \in \{0,1\}$ $e \in E$.

$$x(e) = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{otherwise} \end{cases}$$

Integer Program:

$$\max \sum_{e \in E} w(e)\, x(e)$$

$$v \in V: \quad \sum_{e \in \delta(v)} x(e) \le 1 \quad \leftarrow \text{ at most one edge in } \delta(v) \text{ chosen}$$



$$x \ge 0$$
$$x \in \mathbb{Z}^{|E|}$$

LP relaxation: replace by $x \in \mathbb{R}^{|E|}$

Important principle:

if $x^* \in \mathbb{Z}^{|E|}$ is optimal sol. to LP-relaxation,

then $x^* \in \{0,1\}^{|E|}$ is representing max. weight matching.

Example:

$w = 1$ for all edges.

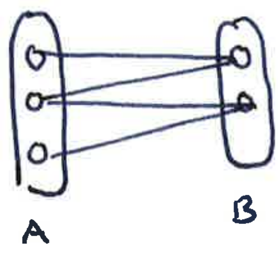
VALUE LP: $3/2$

VALUE IP: 1.

Definition: $G = (V, E)$ is bipartite, if $V = A \,\dot\cup\, B$

and $E \subseteq \{ab : a \in A, b \in B\}$.

Example:



bipartite.

not bipartite.

Lemma: $G = (V, E)$ is bipartite $\Leftrightarrow$ G has no odd cycle.

Proof: Exercise. "$\Rightarrow$"

"$\Leftarrow$"

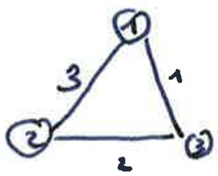
BFS tree.

Node-edge incidence matrix:

$G = (V, E)$ graph

$A_G \in \{0,1\}^{|V| \times |E|}$ $\qquad (A_G)_{v,e} = \begin{cases} 1 & v \text{ endpoint of } e \\ 0 & \text{otherwise.} \end{cases}$

Example:



$$A_G = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\det(A_G) = 2$

Theorem: Let $A \in \mathbb{Z}^{n \times n}$ non-singular. The system

$$Ax = b \quad \checkmark \quad \cancel{b \in \mathbb{Z}^n}$$

has integer solution x for all $b \in \mathbb{Z}^n$ $\iff$ $\det(A) = \pm 1$.

Proof: "$\Leftarrow$" $\quad A^{-1} = \dfrac{\tilde{A}}{\det(A)} \qquad \tilde{A} = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ & \vdots & & \\ A_{1n} & & \cdots & A_{nn} \end{pmatrix}$

$$A_{ij} = (-1)^{i+j} \det \begin{array}{c} \left(\begin{array}{ccc} & | & \\ - & A & - \\ & | & \end{array} \right) i \\ j \end{array}$$

crossed out.

$\tilde{A} \in \mathbb{Z}^{n \times n}$

$\Rightarrow A^{-1} = \dfrac{\tilde{A}}{\pm 1} \in \mathbb{Z}^{n \times n}$

„ $\Rightarrow$ "

$$A \cdot x_i = e_i \quad , i = 1, \dots, n$$

has int. sol. $x_i \in \mathbb{Z}^n$

$$\Rightarrow \quad A \, \underbrace{(x_1 \cdots x_n)}_{= A^{-1} \in \mathbb{Z}^{n \times n}} = I_n$$

$$1 = \underbrace{\det(A)}_{\in \mathbb{Z}} \cdot \underbrace{\det(A^{-1})}_{\in \mathbb{Z}} \quad \Rightarrow \quad \det(A) = \pm 1 .$$

<u>Definition:</u> $A \in \{0, \pm 1\}^{m \times n}$ is called <u>totally unimodular</u> if $\forall \; I \subseteq \{1, \dots, m\}, \; J \subseteq \{1, \dots, n\}$ with $|I| = |J|$, one has

$$\det(A_{IJ}) = \underline{\pm 1, 0}$$

where $(A_{IJ})_{k,l} = (A)_{i_k j_l}$ $\quad I = \{i_1, \dots, i_0\}$, $J = \{j_1, \dots, j_0\}$

in other words each $k \times k$ sub-determinant of A is $\underline{\pm 1, 0}$.

<u>Example:</u> $G = (A + B, E)$ bipartite. A_G node-edge incidence matrix of G.

<u>Theorem:</u> A_G is totally unimodular.

proof:

$$A_G = \begin{array}{c|c} & e \\ A & 1 \\ \hline B & 1 \end{array}$$

Let $C \in \{0,1\}^{k \times k}$ be a $k \times k$ sub-matrix of A_G.

Case 1: Each column has exactly two 1's.

$$C = \begin{array}{c} A \\ B \end{array} \left[\begin{array}{c} 1 \\ \hline 1 \end{array} \right] \qquad \det(C) = 0$$

Case 2: one col. has no 1 : $\det(C) = 0$

Case 3: one col. has exactly one 1 :

$$C \quad \begin{array}{c} A \\ B \end{array} \left[\begin{array}{c} 1 \\ \hline \end{array} \right] \quad \text{develop}$$

$\det(C) = \pm 1 \cdot \det(C')$ $\qquad$ C' $(k-1) \times (k-1)$ sub-matrix.

Induction

$\blacksquare$

Theorem: (Hoffman-Kruskal)

Let $A \in \mathbb{Z}^{m \times n}$. $P = \{x \in \mathbb{R}^n : Ax \le b,\ x \ge 0\}$ is integral for each $b \in \mathbb{Z}^m$ $\Leftrightarrow$ A is totally unimodular.

proof: We do only „$\Leftarrow$".

to show: Each feasible basis yields integer vertex.

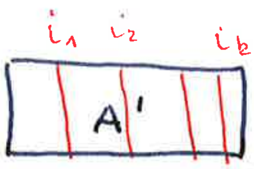
$$P: \quad \begin{array}{rl} A\,x & \le b \\ -I\,x & \le 0 \end{array} \quad \approx \quad \begin{pmatrix} A \\ -I \end{pmatrix} x \le \begin{pmatrix} b \\ 0 \end{pmatrix}$$

$B \subseteq \{1, \dots, m+n\}$ ~~basis~~ basis

$$A_B = \left[\begin{array}{c} A' \\ \hline -e_{i_1}^T \\ \vdots \\ -e_{i_k}^T \end{array}\right]$$

develop determinant

~~After~~ $\det(A_B) = \pm \det(A^*)$

A^* as 

columns deleted.

A^* is $(n-k) \times (n-k)$ submatrix of A $\Rightarrow$ $\det(A_B) = \pm 1$

Basic solution: $x^*_B = \underbrace{A_B^{-1} \cdot b_B}_{\in \mathbb{Z}^n}$

$\blacksquare$

Vertex Cover: $G=(V,E)$ $\quad b: V\to\mathbb{R}$.

$S\subseteq V$ is vertex cover if

$$\forall e\in E: \ e\cap S \neq \emptyset$$

Min-weight vertex cover:

$$\min \sum_{v\in V} x(v)\cdot b(v)$$

$$uv\in E: \quad x(u)+x(v)\geq 1$$

$$x\geq 0$$

$$x\in\mathbb{Z}^{|V|}$$

$$\min \sum_{v\in V} x(v)\cdot b(v)$$

if G is bipartite, then A_G^T is TU.

$$A_G^T\cdot x\geq \mathbb{1}$$

$$x\geq 0$$

$$x\in\mathbb{Z}^{|V|}$$

Theorem: (König's Theorem)

In a bipartite graph, the number of edges in a max. matching equals the number of vertices in a min-vertex cover.

Proof:

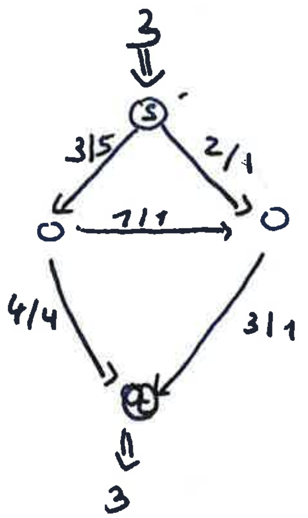
$$\max \ \mathbb{1}^T\cdot x \qquad\qquad \min \ \mathbb{1}^T\cdot y$$

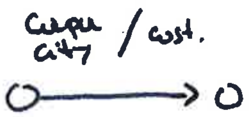
$$A_G x\leq 1 \qquad\qquad A_G^T\cdot y\geq 1$$

$$x\geq 0 \qquad\qquad y\geq 0$$

DUALS

Flows:





Route flow ~~of~~ at min-cost

Definition (Flow): Let $D = (V, A)$ be directed graph

$u: A \to \mathbb{R}^+$ capacities

$c: A \to \mathbb{R}$ cost.

$s \neq t \in V$ designated nodes and $z \in \mathbb{N}$ value.

Min-cost s-t flow.

$$\min \sum_{e \in A} f(e) \cdot c(e)$$

$$v \in V \setminus \{s,t\} \qquad \sum_{e \in \delta^+(v)} f(e) = \sum_{e \in \delta^-(v)} f(e)$$

$$0 \le f(e) \le u(e) \quad \forall e \in A$$

$$\vdots$$