École Polytechnique Fédérale de Lausanne
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Discrete Optimization (Spring 2025)

Assignment 9

1) Sudoku is the following puzzle: Given a matrix

$$A \in \{1, \ldots, 9, X\}^{9\times 9}$$

the task is to replace each X in A by a number in $\{1, \ldots, 9\}$ such that the following holds.

a) Each line of A contains all numbers in $\{1, \ldots, 9\}$

b) Each column of A contains all numbers in $\{1, \ldots, 9\}$

c) If A is written as

$$A = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$$

where the B_{ij} are 3×3 matrices, the each of them contains all numbers in $\{1, \ldots, 9\}$.

In the following, you are asked to design an integer program whose feasible solutions are solutions of a given Sudoku. The integer program has 9^3 variables

$$x_{i,j,k} = \begin{cases} 1 & \text{if } s_{i,j} = k \\ 0 & \text{otherwise.} \end{cases}$$

where $S \in \{0, \ldots, 9\}^{9\times 9}$ is the solution of the Sudoku represented by the variables.
The following set of constraints guarantees that each cell of the solution contains a number

$$1 \leqslant i, j \leqslant 9 : \quad \sum_{k=1}^{9} x_{i,j,,k} = 1.$$

Write now constraints that guarantee the following.

- Each row must contain each number.
- Each column must contain each number.
- Each B_{ij} must contain each number.

Describe the final integer programming problem that links some of the variables to the input Sudoku.

2) Let $a, b \in \mathbb{Z}$ not both equal to zero. Describe an integer program with variables $x, y \in \mathbb{Z}$ whose optimal value is the greatest common divisor of a and b and with optimal solution $x, y \in \mathbb{Z}$ being the corresponding Bézout-coefficients

$$\gcd(a, b) = x \cdot a + y \cdot b.$$

3) Consider the polyhedron $P = \{x \in \mathbb{R}^3 \colon x_1 + 2\,x_2 + 4\,x_3 \leqslant 4,\ x \geqslant 0\}$. Show that this polyhedron is integral.

4) *(Clustering)* The following is a recurring problem in data science. Suppose we are given m points $v_1, \ldots, v_m \in \mathbb{R}^n$ and a number k. We want to identify a subset $S \subseteq \{v_1, \ldots, v_m\}$ of size k such that

$$\sum_{i=1}^{m} d(v_i, S) \text{ is minimal.}$$

Here $d(v, S)$ is the euclidean distance of v to S. Write an integer program that models this problem.

5) Show the following: A polyhedron $P \subseteq \mathbb{R}^n$ with vertices is integral, if and only if each vertex is integral.