École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

April 15, 2025

Discrete Optimization (Spring 2025)

Assignment 9

1) Sudoku is the following puzzle: Given a matrix

$$A \in \{1, \ldots, 9, X\}^{9\times 9}$$

the task is to replace each X in A by a number in $\{1, \ldots, 9\}$ such that the following holds.

a) Each line of A contains all numbers in $\{1, \ldots, 9\}$

b) Each column of A contains all numbers in $\{1, \ldots, 9\}$

c) If A is written as

$$A = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$$

where the B_{ij} are 3×3 matrices, the each of them contains all numbers in $\{1, \ldots, 9\}$.

In the following, you are asked to design an integer program whose feasible solutions are solutions of a given Sudoku. The integer program has 9^3 variables

$$x_{i,j,k} = \begin{cases} 1 & \text{if } s_{i,j} = k \\ 0 & \text{otherwise.} \end{cases}$$

where $S \in \{0, \ldots, 9\}^{9\times 9}$ is the solution of the Sudoku represented by the variables.
The following set of constraints guarantees that each cell of the solution contains a number

$$1 \leqslant i, j \leqslant 9 : \quad \sum_{k=1}^{9} x_{i,j,,k} = 1.$$

Write now constraints that guarantee the following.

- Each row must contain each number.
- Each column must contain each number.
- Each B_{ij} must contain each number.

Describe the final integer programming problem that links some of the variables to the input Sudoku.

Solution:
The condition that each row must contain each number, corresponds to the set of constraints

$$\sum_{j=1}^{9} x_{i,j,k} = 1, \quad \forall\, 1 \leqslant i, k \leqslant 9.$$

The condition that each column must contain each number, corresponds to the set of constraints

$$\sum_{i=1}^{9} x_{i,j,k} = 1, \quad \forall\, 1 \leqslant j, k \leqslant 9.$$

The condition that each B_{ij} must contain each number, corresponds to the set of constraints

$$\sum_{\alpha=0}^{2}\sum_{\beta=0}^{2} x_{i+\alpha,j+\beta,k} = 1, \quad \forall\, i,j = 1,4,7,\ 1 \leqslant k \leqslant 9.$$

To get the final integer program, we only need the set of constraints which guarantees that the solution is compatible with the input, i.e.,

$$x_{i,j,k} = 1, \quad \text{if in the input of Sudoku, the cell } (i,j) \text{ is assigned the number } k$$

and the integrality constraints

$$x_{i,j,k} \in \{0,1\}, \quad \forall\, 1 \leqslant i,j,k \leqslant 9.$$

2) Let $a, b \in \mathbb{Z}$ not both equal to zero. Describe an integer program with variables $x, y \in \mathbb{Z}$ whose optimal value is the greatest common divisor of a and b and with optimal solution $x, y \in \mathbb{Z}$ being the corresponding Bézout-coefficients

$$\gcd(a,b) = x \cdot a + y \cdot b.$$

Solution:
Consider the following integer program:

$$\begin{aligned} \min \quad & ax + by \\ s.t. \quad & x, y \in \mathbb{Z} \\ & ax + by \geqslant 1 \end{aligned}$$

We claim that the optimal solution to the integer program above is the greatest common divisor of a and b. Notice that the integer program is feasible since a, b are not both equal to zero. Let d be the optimal solution of the integer program, with optimal solution x, y. It is easy to see that $\gcd(a,b) \mid d$. To show $d \mid \gcd(a,b)$, it suffices to show that $d \mid a$ and $d \mid b$. Suppose that $d \nmid a$. Then there exists $q, r \in \mathbb{Z}, 1 \leqslant r < d$ such that $a = qd + r$. Therefore we have $r = a - qd = a - q(ax + by) = (1 - qx)a + (-qy)b$, which contradicts the optimality of d. Similarly one can prove that $d \mid b$.

3) Consider the polyhedron $P = \{x \in \mathbb{R}^3 \colon x_1 + 2\,x_2 + 4\,x_3 \leqslant 4,\ x \geqslant 0\}$. Show that this polyhedron is integral.

Solution:
By the result of Exercise 5, it suffices to show that each vertex of P is integral. The system of constraints are

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &\leqslant 4, \\ x_1 &\geqslant 0, \\ x_2 &\geqslant 0, \\ x_3 &\geqslant 0. \end{aligned}$$

Hence the 4 vertices of P are $(0,0,0), (4,0,0), (0,2,0), (0,0,1)$.

4) *(Clustering)* The following is a recurring problem in data science. Suppose we are given m points $v_1, \dots, v_m \in \mathbb{R}^n$ and a number k. We want to identify a subset $S \subseteq \{v_1, \dots, v_m\}$ of size k such that

$$\sum_{i=1}^{m} d(v_i, S) \text{ is minimal.}$$

Here $d(v, S)$ is the euclidean distance of v to S. Write an integer program that models this problem.

Solution:
We assume that all the pairwise Euclidean distance $d_{i,j}$ between any points $v_i, v_j, 1 \leqslant i, j \leqslant m$ have been pre-computed. Let y_j be the decision variable indicating whether the point v_j is chosen to be in S. Let $x_{i,j}$ to be the variable indicating whether the point v_j is the closest point to v_i in S. Consider the following integer program, which models the clustering problem:

$$\begin{aligned}
\min \quad & \sum_{i=1}^{m}\sum_{j=1}^{m} d_{i,j}x_{i,j} \\
s.t. \quad & \sum_{j=1}^{m} x_{i,j} = 1, \quad \forall 1 \leqslant i \leqslant m \\
& x_{i,j} \leqslant y_j, \quad \forall 1 \leqslant i, j \leqslant m \\
& \sum_{j=1}^{m} y_j = k, \\
& x_{i,j}, y_j \in \{0, 1\}, \quad \forall 1 \leqslant i, j \leqslant m
\end{aligned}$$

5) Show the following: A polyhedron $P \subseteq \mathbb{R}^n$ with vertices is integral, if and only if each vertex is integral.

Solution:
The "only if" part is easy since each vertex is itself a face of P. To show the "if" part, assume that each vertex of P is integral. Let $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$. Consider any nonempty face F of P, which is of the form $F = P \cap \{x \in \mathbb{R}^n : c^T x = \beta\}$ where $c^T x \leqslant \beta$ is valid for P.

- A proof for P being a **bounded** polyhedron: If $P = \text{conv}(\text{Ver}(P))$ where $\text{Ver}(P)$ is the set of all vertices of P, then we take any $x \in F$ which can be written as $x = \sum_{i=1}^{k} \alpha_i v_i$ where v_i is a vertex of P and $\alpha_i > 0, \sum_{i=1}^{k} \alpha_i = 1$. We claim that each v_i in the expression above, which is integral by assumption, is in the face F. Indeed, we have

$$\beta = c^T x = c^T \left(\sum_{i=1}^{k} \alpha_i v_i \right) = \sum_{i=1}^{k} \alpha_i (c^T v_i) \leqslant \sum_{i=1}^{k} \alpha_i \beta = \beta,$$

  which implies that every inequalities inside are in fact equalities. Thus for each i we have $c^T v_i = \beta$, hence $v_i \in F$.

- A proof for general polyhedron P: If P has vertices, then $\text{rank}(A) = n$. Thus for the face F which is itself a polyhedron characterized by $Ax \leqslant b, c^T x \leqslant \beta, -c^T x \leqslant -\beta$, its constraint matrix has rank n, which means that F also contains a vertex. We just need to show that a vertex of F is also a vertex of P. To prove this, we first claim that there exists

$u \in \mathbb{R}^m, u \geqslant 0$ such that $u^T A = c^T$ and $u^T b = \beta$. Indeed, consider the following pair of primal and dual linear programs:

$$\begin{array}{ccc} \max c^T x & \quad\text{and}\quad & \min b^T y \\ Ax \leqslant b & & y^T A = c^T \\ & & y \geqslant 0 \end{array}$$

Since $c^T x \leqslant \beta$ is valid for P, we know that P is bounded. Also P is feasible since it has vertices. The optimal value of the primal problem is β, since the face F is nonempty. Then by the strong duality theorem, the dual problem has an optimal solution, say $u \in \mathbb{R}^m, u \geqslant 0$ with $u^T A = c^T$, such that the value is also β, i.e., $u^T b = \beta$. Then we have $F = P \cap \{x \in \mathbb{R}^n : A'x = b'\}$ where $A'x \leqslant b'$ is a sub-system of $Ax \leqslant b$ by taking the constraint $a_i^T x \leqslant b_i$ of $Ax \leqslant b$ if and only if $u_i > 0$. The fact $F = P \cap \{x \in \mathbb{R}^n : A'x = b'\}$ implies that vertices of F are also vertices of P.