École Polytechnique Fédérale de Lausanne
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Discrete Optimization (Spring 2025)

Assignment 7

1) Consider the following problem. We are given $B \in \mathbb{N}$, and a set of integer points $S = \{p \in \mathbb{Z}^n : 0 \leqslant p_i \leqslant B \forall i = 1, ..., n\}$, whose points are all colored blue but one, which is red. We have an oracle that, given vectors $l, r \in \mathbb{R}^n$, tells us whether the red point in S is contained in the box $S \cap \{x \in \mathbb{R}^n : l_i \leqslant x_i \leqslant r_i \forall i = 1, ..., n\}$ or not. Give an algorithm to find the red point using $O(n \log(B))$ many oracle calls.

2) Let $P := \{x \in \mathbb{R}^n : Ax = b, x \geqslant 0\}$ be a polyhedron and $\min\{c^T x : x \in P\}$ be the corresponding primal linear program. Assume that all the coefficients of A, b and c are integral and bounded in absolute value by given $B \in \mathbb{N}$, and furthermore let $L := B^n n^{n/2}$.

   (a) Show the following: If x_1, x_2 are vertices of P and $c^T x_1 \neq c^T x_2$, then $|c^T x_1 - c^T x_2| \geqslant 1/L^2$.

   (b) Let $x^*$and y^* be feasible solutions of the primal and dual linear program respectively. Conclude the following from the above: If $|c^T x^* - b^T y^*| < 1/L^2$, then each vertex x of P with $c^T x \leqslant c^T x^*$ is an optimal solution of the primal.

3) Let $Ax \leqslant b$ be a system of inequalities where each component of A and b is an integer bounded by B in absolute value. Show that $Ax \leqslant b$ is feasible if and only if $Ax \leqslant b, -B^n \cdot n^{n/2} \cdot n \cdot B \leqslant x_i \leqslant B^n \cdot n^{n/2} \cdot n \cdot B$, $\forall i \in [n]$ is feasible.

   Hint: Consider a feasible point x^* and the index sets $I = \{i : x_i^* \geqslant 0\}$ and $J = \{j : x_j^* \leqslant 0\}$. The polyhedron defined by $Ax \leqslant b, x_i \geqslant 0, i \in I, x_j \leqslant 0, j \in J$ is feasible and has vertices. Estimate the infinity norm of a vertex.

4) Suppose that there exists an algorithm that on input $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ decides the feasibility of the system $Ax \leqslant b$, in time $poly(n, m, \log B)$, where B is an upper bound on each absolute value of an entry of A and b.

   Let the system $P = \{Ax \leqslant b\}$ be feasible where P contains vertices. Let $c \in \mathbb{Z}^n$ such that $\max\{c^T x : Ax \leqslant b\} < \infty$ and $\|c\|_\infty \leqslant B$. Using binary search, show that there exists a polynomial time (in n, m and $\log B$) algorithm that on input A, b, c determines the value of $\max\{c^T x : Ax \leqslant b\}$.