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Discrete Optimization (Spring 2025)

Assignment 7

1) Consider the following problem. We are given $B \in \mathbb{N}$, and a set of integer points $S = \{p \in \mathbb{Z}^n : 0 \leqslant p_i \leqslant B \forall i = 1, ..., n\}$, whose points are all colored blue but one, which is red. We have an oracle that, given vectors $l, r \in \mathbb{R}^n$, tells us whether the red point in S is contained in the box $S \cap \{x \in \mathbb{R}^n : l_i \leqslant x_i \leqslant r_i \forall i = 1, ..., n\}$ or not. Give an algorithm to find the red point using $O(n \log(B))$ many oracle calls.

Solution:
We split the problem in n subproblems: for $i \in \{1, ..., n\}$, we want to obtain the i-th component of the red point. This is a simple binary search problem, and we illustrate how to solve it for $i = 1$: we first call the oracle with $l = (0, ..., 0), r = (B/2, B, ..., B)$. Then we call the oracle with $l = (0, ..., 0), r = (B/4, B, ..., B)$ (if the answer is positive) or $l = (B/2, ..., 0), r = (3B/2, B, ..., B)$ (if the answer is negative), and so on. In this way, we are guaranteed to find each component of the red point with $O(\log(B))$ oracle calls, for a total of $O(n \log(B))$ many oracle calls.

2) Let $P := \{x \in \mathbb{R}^n : Ax = b, x \geqslant 0\}$ be a polyhedron and $\min\{c^T x : x \in P\}$ be the corresponding primal linear program. Assume that all the coefficients of A, b and c are integral and bounded in absolute value by given $B \in \mathbb{N}$, and furthermore let $L := B^n n^{n/2}$.

   (a) Show the following: If x_1, x_2 are vertices of P and $c^T x_1 \neq c^T x_2$, then $|c^T x_1 - c^T x_2| \geqslant 1/L^2$.

   (b) Let $x^*$and y^* be feasible solutions of the primal and dual linear program respectively. Conclude the following from the above: If $|c^T x^* - b^T y^*| < 1/L^2$, then each vertex x of P with $c^T x \leqslant c^T x^*$ is an optimal solution of the primal.

Solution:

   (a) Let B_1 and B_2 be the sets of row indices corresponding to basic feasible solutions x_1, x_2, then $x_1 = \begin{pmatrix} A \\ I \end{pmatrix}_{B_1}^{-1} \begin{pmatrix} b \\ 0 \end{pmatrix}_{B_1}$ and $x_2 = \begin{pmatrix} A \\ I \end{pmatrix}_{B_2}^{-1} \begin{pmatrix} b \\ 0 \end{pmatrix}_{B_2}$. A basic result from linear algebra states that given an $n \times n$ invertible matrix M one has that $M^{-1} = \frac{1}{\det(M)} adj(M)$. If one has that the entries of M are integral and bounded in absolute value by B then Hadamard's bound gives that $\det(M) \leqslant B^n n^{n/2} = L$. Combining the two statements gives that the entries of M^{-1} are integer multiples of $1/|det(M)| \geqslant 1/L$. Applying this statement to $\begin{pmatrix} A \\ I \end{pmatrix}_{B_1}$ and since b is integral we obtain that each entry of x_1 is an integer multiple of $1/\left|\det\left(\begin{pmatrix} A \\ I \end{pmatrix}_{B_1}\right)\right| \geqslant 1/L$. Analogously obtain the same bound for x_2. Thus we obtain that $|c^T x_1 - c^T x_2| = |c^T (x_1 - x_2)|$ is a positive integer multiple of some $\delta \geqslant 1/L^2$ since $c \in \mathbb{Z}^n$.

   (b) We prove the statement by contradiction. Assume that x is not an optimal solution, then there is a vertex $\bar{x}$ such that $c^T \bar{x} < c^T x$. Then by weak duality one has $b^T y^* \leqslant c^T \bar{x} < c^T x \leqslant c^T x^*$ which further implies $|c^T \bar{x} - c^T x| < 1/L^2$ contradicting part (a).

3) Let $Ax \leqslant b$ be a system of inequalities where each component of A and b is an integer bounded by B in absolute value. Show that $Ax \leqslant b$ is feasible if and only if $Ax \leqslant b, -B^n \cdot n^{n/2} \cdot n \cdot B \leqslant x_i \leqslant B^n \cdot n^{n/2} \cdot n \cdot B, \forall i \in [n]$ is feasible.

Hint: Consider a feasible point x^* and the index sets $I = \{i : x_i^* \geqslant 0\}$ and $J = \{j : x_j^* \leqslant 0\}$. The polyhedron defined by $Ax \leqslant b, x_i \geqslant 0, i \in I, x_j \leqslant 0, j \in J$ is feasible and has vertices. Estimate the infinity norm of a vertex.

Solution:
By using the hint, we obtain a system of inequalities with matrix $\bar{A}$ of full column rank and vector $\bar{b}$. Indeed, one can see that in the new system $\bar{A}x \leqslant \bar{b}$, $\bar{A}$ is of the form

$$\bar{A} = \begin{pmatrix} A \\ X \\ Y \end{pmatrix}$$

and $\bar{b} = (b, 0)$, where X is an $|I| \times n$ matrix, where the x_{kl} entry is equal to 1 if $k = l$ and $k \in I$, Y is a $|J| \times n$ matrix where the y_{kl} entry is equal to -1 if $k = l$ and $k \in J$ and 0 otherwise. By construction of $\bar{A}$, we can see that the matrix has full column rank. This implies that the corresponding polyhedron has vertices. Moreover, by a theorem from the course, a vertex is characterized by a subsystem $A'x \leqslant b'$ where $rank(A') = n$ such that $x^* = A'^{-1}b'$.

We know from linear algebra that $A^{-1} = adj(A')/\det(A')$. Hence,

$$|x_i^*| = \left| \sum_{j=1}^{n} a'^{-1}_{ij} b_j \right| \leqslant B \sum_{j=1}^{n} \left| a'^{-1}_{ij} \right|.$$

Then we just have to bound the entries of the matrix A'^{-1}. From the fact that A' is invertible and has integer entries, we get that $|\det(A')| \geqslant 1$. From Hadamard's bound, we also have that $|\det(adj(A))| \leqslant B^{n-1}(n-1)^{\frac{n-1}{2}}$. Combining these two facts, we have the desired result,

$$|x_i^*| \leqslant B \sum_{j=1}^{n} \left| a'^{-1}_{ij} \right| \leqslant B \sum_{j=1}^{n} B^{n-1}(n-1)^{\frac{n-1}{2}} \leqslant n \cdot B \cdot B^{n-1}(n-1)^{\frac{n-1}{2}} \leqslant n \cdot B \cdot n^{n/2} \cdot B^n.$$

4) Suppose that there exists an algorithm that on input $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ decides the feasibility of the system $Ax \leqslant b$, in time $poly(n, m, \log B)$, where B is an upper bound on each absolute value of an entry of A and b.

Let the system $P = \{Ax \leqslant b\}$ be feasible where P contains vertices. Let $c \in \mathbb{Z}^n$ such that $\max\{c^T x : Ax \leqslant b\} < \infty$ and $\|c\|_\infty \leqslant B$. Using binary search, show that there exists a polynomial time (in n, m and $\log B$) algorithm that on input A, b, c determines the value of $\max\{c^T x : Ax \leqslant b\}$.

Solution:
From the assumptions, there exists a vertex x^* of P such that $c^T x^* = \max\{c^T x : Ax \leqslant b\}$. We have that there is some invertible A', submatrix of A, and a b' subvector of b, such that $x^* = A'^{-1}b'$, hence $|x_i^*| = \sum_{j=1}^{m} \frac{|A'_{ij}|}{|\det(A')|} b'_j$ and using Hadamard's bound and the integrality of A, b, c, we get that

$$|c^T x^*| \leqslant m B^{n+2} n^{n/2+1}.$$

Moreover, from exercise 2, we have that for any non-optimal vertex x of P, $c^T x^* - c^T x \geqslant 1/L^2$, where $L = B^n n^{n/2}$. Hence we need to find an interval $[\alpha, \beta]$ such that $P' := P \cap \{x \in \mathbb{R}^n :$

$c^T x \geqslant \alpha\}$ is feasible and $P'' := P \cap \{x \in \mathbb{R}^n : c^T x \geqslant \beta\}$ is not feasible, and $\beta - \alpha < 1/L^2$. This will then give the value of the maximum. Checking feasibility of these two polytopes is then done in polynomial time in $n, m \log B$ by the assumption that there exists an algorithm that decides feasibility. To find the desired α, β, we perform binary search on the interval $[-mB^{n+2}n^{n/2+1}, mB^{n+2}n^{n/2+1}]$, halving the size of the interval iteratively until the interval has small enough length (less than $1/L^2$). Formally, denote by S_i the system $Ax \leqslant b$, $c^T x \geqslant \alpha_i$ and set $\alpha_0 = -mB^{n+2}n^{n/2+1}, \beta_0 = mB^{n+2}n^{n/2+1}$. At the i-th iteration, we know that S_{i-1} is feasible and we ask to the oracle whether S_i is feasible, with $\alpha_i = \alpha_{i-1} + (\beta_i - \alpha_i)/2$. If it is feasible, we set $\beta_i = \beta_{i-1}$, otherwise we set $\beta_i = \alpha_i, \alpha_i = \alpha_{i-1}$, and we move to the next iteration, until $\beta_i - \alpha_i < 1/L^2$. At this point, if the system $Ax \leqslant b, c^T x \geqslant \beta_i$ is not feasible, we obtained our α, β, otherwise notice that we must have $\beta_i = \beta_0$ which implies that $\max\{c^T x : Ax \leqslant b\} = \beta_0$. The total number of iterations is at most $\log(L^2 \cdot 2mB^{n+2}n^{n/2+1})$, for a total running time that is polynomial in n, m and $\log B$.