École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

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Discrete Optimization (Spring 2025)

Assignment 6

1) Determine the value of the matrix game defined by

$$A = \begin{pmatrix} 6 & 6 \\ 7 & 4 \end{pmatrix}$$

and determine optimal strategies for both players with

(a) pure strategies and

(b) mixed strategies.

2) This exercise is a continuation of exercise 2) from the sheet of last week. Here we find the *Chebychev center* of a polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. This is the center $z \in \mathbb{R}^n$ of the largest euclidean ball $B(z, R) = \{x \in \mathbb{R}^n \colon \|x - z\|_2 \leqslant R\}$ that satisfies $B(z, R) \subseteq P$.

i) Let $H = (a^T x = \beta) \subseteq \mathbb{R}^n$ be a hyperplane and $x^* \in \mathbb{R}^n$. What is the *euclidean distance* of x^* from H?

ii) Assume now that every row of A has euclidean norm $\| \cdot \|_2$ equal to one. Prove that the following linear program finds the Chebychev center z and the radius $R \in \mathbb{R}_{\geqslant 0}$ of the largest ball $B(z, R) \subseteq P$:

$$\begin{array}{c} \max R \\ Az + \mathbf{1}R \leqslant b \end{array},$$

and $\mathbf{1} \in \mathbb{R}^m$ is the vector of all ones.

iii) Show that there is a subsystem $A'x \leqslant b'$ of $Ax \leqslant b$ with at most $n + 1$ inequalities whose corresponding polyhedron has the same Chebychev center as P.

iv) Write down the dual of the linear program above.

3) (Complementary slackness)

Consider the primal/dual pair

$$\begin{array}{c} \max c^T x \\ Ax \leqslant b \end{array} \qquad \text{and} \qquad \begin{array}{c} \min b^T y \\ y^T A = c^T \\ y \geqslant 0 \end{array}$$

defined by $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Let $x^* \in \mathbb{R}^n$ and $y^* \in \mathbb{R}^m$ be feasible primal and dual solutions respectively.

Show the following: x^* and y^* are both optimal solutions respectively if and only if $y_i^* > 0 \implies A_i x^* = b_i$ for each $i \in [m]$.

4) Consider the linear programming problems

$$\begin{array}{c} \max c^T x \\ Ax \leqslant b \\ x \geqslant 0 \end{array} \qquad \text{and} \qquad \begin{array}{c} \min b^T y \\ y^T A \geqslant c^T \\ y \geqslant 0 \end{array}$$

i) Show that the minimization problem on the right is equivalent to the dual of the maximization problem.

ii) Let x^* and y^* be feasible solutions of the maximization and minimization problem respectively. Show that they are both optimal solutions respectively if and only if the following condition holds:

$$(y^*)^T(b - Ax^*) = 0 \text{ and } (y^T A - c^T)x^* = 0.$$