École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

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Discrete Optimization (Spring 2025)

Assignment 6

1) Determine the value of the matrix game defined by

$$A = \begin{pmatrix} 6 & 6 \\ 7 & 4 \end{pmatrix}$$

and determine optimal strategies for both players with

(a) pure strategies and
(b) mixed strategies.

Solution:

(a) For row player, the best pure strategy is to choose row 1 since the minimum value for row 1 is 6 which is higher than the minimum value for row 2. Similarly, the best strategy for column player is to choose column 2.

(b) The row player chooses x such that $\sum_i x_i = 1$ and x is the solution of $\max_x \min_y x^T Ay$. Note that for any vector $x \geqslant 0$, the minimizing vector y such that $y_1 + y_2 = 1$ is given by $y = (0, 1)$. Then we have that $x^T Ay = 6x_1 + 4x_2$ such that the maximizing vector x is $(1, 0)$. This gives the optimal mixed strategy which is actually a pure strategy.

The value of the matrix game is thus 6.

2) This exercise is a continuation of exercise 2) from the sheet of last week. Here we find the *Chebychev center* of a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. This is the center $z \in \mathbb{R}^n$ of the largest euclidean ball $B(z, R) = \{x \in \mathbb{R}^n : \|x - z\|_2 \leqslant R\}$ that satisfies $B(z, R) \subseteq P$.

i) Let $H = (a^T x = \beta) \subseteq \mathbb{R}^n$ be a hyperplane and $x^* \in \mathbb{R}^n$. What is the *euclidean distance* of x^* from H?

ii) Assume now that every row of A has euclidean norm $\|\cdot\|_2$ equal to one. Prove that the following linear program finds the Chebychev center z and the radius $R \in \mathbb{R}_{\geqslant 0}$ of the largest ball $B(z, R) \subseteq P$:

$$\begin{array}{c} \max R \\ Az + \mathbf{1}R \leqslant b \end{array},$$

and $\mathbf{1} \in \mathbb{R}^m$ is the vector of all ones.

iii) Show that there is a subsystem $A'x \leqslant b'$ of $Ax \leqslant b$ with at most $n + 1$ inequalities whose corresponding polyhedron has the same Chebychev center as P.

iv) Write down the dual of the linear program above.

Solution:

i) The distance of x^* to H is the distance of x^* to the orthogonal projection onto H. This is given by $\|x^* - proj_H(x^*)\| = \frac{|\beta - \langle x^*, \alpha\rangle|}{\|\alpha\|}$.

ii) Let z^* be the Chebyshev center and R^* be the corresponding radius. The ball $B(z^*, R^*) = \{x \in \mathbb{R}^n : \|x - z^*\| \leqslant R^*\} = \{z^* + x : \|x\| \leqslant R^*\}$. Then the Chebyshev center is defined by

$$\max_{z^*, R^*} R^*$$
$$\text{s.t. } B(z^*, R^*) \subseteq P.$$

This. is equivalent to

$$\max_{z^*, R^*} R^*$$
$$\text{s.t. } A(z^* + x) \leqslant b \qquad \text{for every } x \in \mathbb{R}^n \text{ s.t. } \|x\| \leqslant R^*.$$

Note that for any row i, $a_i^T(z^* + x) \leqslant a_i^T(z^* + a_i/\|a_i\|R^*)$ since $\|x\| \leqslant R^*$ and $a_i^T x \leqslant \|x\| \frac{a_i^T a_i}{\|a_i\|}$ (this is just the projection from part (i)). Then, the Chebyshev center is defined by

$$\max_{z^*, R^*} R^*$$
$$\text{s.t. } a_i^T(z^* + a_i/\|a_i\|R^*) \leqslant b_i \quad i \in [m]$$

since if the above constraints hold, then the constraints over all x with $\|x\| \leqslant R^*$ also hold. Then this LP is equivalent to

$$\max_{z^*, R^*} R^*$$
$$\text{s.t. } A^T z^* + 1R^* \leqslant b$$

since $\|a_i\| = 1$ for every i.

iii) The vector $(z^*, R^*) \in \mathbb{R}^{n+1}$ is the maximizer of the LP given in part (ii). Then there exist a set of at most $n+1$ tight inequalities of the LP so that the subsystem of these inequalities uniquely define the optimizer (z^*, R^*). Then the polyhedron for this subsystem also has (z^*, R^*) as an optimizer such that this system defines the same Chebyshev center.

iv) The dual linear program is

$$\begin{aligned} &\min b^T y \quad , \\ &A^T y = \mathbf{0} \\ &\mathbf{1}^T y = 1 \\ &y \geqslant 0 \end{aligned}$$

3) (Complementary slackness)

Consider the primal/dual pair

$$\begin{array}{ccc} \max c^T x & \text{and} & \min b^T y \\ Ax \leqslant b & & y^T A = c^T \\ & & y \geqslant 0 \end{array}$$

defined by $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Let $x^* \in \mathbb{R}^n$ and $y^* \in \mathbb{R}^m$ be feasible primal and dual solutions respectively.

Show the following: x^* and y^* are both optimal solutions respectively if and only if $(y^*)i > 0 \implies A_i x^* = b_i$ for each $i \in [m]$.

Solution:
First we show $\Leftarrow$.

By the assumption, we have that

$$\begin{aligned}
{y^*}^T b &= \sum_{j=1}^{m} y_j^* b_j \\
&= \sum_{j=1}^{m} y_j^* (A_j x^*) \\
&= \sum_{j=1}^{m} y_j^* \left(\sum_{i=1}^{n} A_{ji} x_i^* \right) \\
&= \sum_{i=1}^{n} x_i^* \left(\sum_{j=1}^{m} A_{ji} y_j^* \right) \\
&= {x^*}^T c \\
&= c^T x^*
\end{aligned}$$

so that x^*, y^* achieve the same primal/dual objective value and are therefore optimal by strong duality. For the second equality, we have used that whenever the summand is nonzero, $A_j x^* = b_j$.

We then show $\Rightarrow$.

We have that as x^*, y^* are optimal, $c^T x^* = b^T y^*$ by strong duality. Then

$$\begin{aligned}
c^T x^* &= \sum_{i=1}^{n} x_i^* c_i \\
&= \sum_{i=1}^{n} x_i^* \left(\sum_{j=1}^{m} A_{ji} y_j^* \right) \\
&= \sum_{j=1}^{m} y_j^* \left(\sum_{i=1}^{n} A_{ji} x_i^* \right) \\
&= \sum_{j=1}^{m} y_j^* (A_j x^*) \\
&\leqslant \sum_{j=1}^{m} y_j^* b_j \\
&= {y^*}^T b
\end{aligned}$$

where we have used that $y_j^* \geqslant 0$ for every j in the last inequality. Then as the above inequalities must hold with equality everywhere since $c^T x^* = b^T y^*$, it must be that if $y_j^* > 0$ then $A_j x^* = b_j$.

4) Consider the linear programming problems

$$\begin{array}{ccc}
\max c^T x & \text{and} & \min b^T y \\
Ax \leqslant b & & y^T A \geqslant c^T \\
x \geqslant 0 & & y \geqslant 0
\end{array}$$

i) Show that the minimization problem on the right is equivalent to the dual of the maximization problem.

ii) Let x^* and y^* be feasible solutions of the maximization and minimization problem respectively. Show that they are both optimal solutions respectively if and only if the following condition holds:

$$(y^*)^T(b - Ax^*) = 0 \text{ and } (y^TA - c^T)x^* = 0.$$

Solution:

i) We rewrite the maximization problem as:

$$\begin{gathered}\max c^T x\\ \tilde{A}x \leqslant \tilde{b}\end{gathered}$$

where $\tilde{A} = \begin{pmatrix} A \\ -I \end{pmatrix}$ and $\tilde{b} = \begin{pmatrix} b \\ \mathbf{0} \end{pmatrix}$. Then the dual of this maximization problem is

$$\begin{gathered}\min \tilde{b}^T \tilde{y}\\ \tilde{y}^T \tilde{A} = c^T\\ \tilde{y} \geqslant 0\end{gathered}$$

Let $\tilde{y} = \begin{pmatrix} y \\ y' \end{pmatrix}$ then the objective becomes $\tilde{b}^T\tilde{y} = b^T y$ and the constraint becomes $\tilde{y}^T\tilde{A} = y^TA - (y')^T = c^T, y, y' \geqslant 0 \Leftrightarrow y^TA = c^T, y \geqslant 0$. Therefore the dual problem is equivalent to

$$\begin{gathered}\min b^T y\\ y^T A \geqslant c^T\\ y \geqslant 0\end{gathered}$$

which is the minimization problem on the right.

ii) By weak duality we have

$$c^T x^* \leqslant (y^*)^T Ax^* \leqslant (y^*)^T b.$$

Both x^* and y^* are optimal solutions if and only if both of inequalities above are equalities, which is equivalent to

$$(y^TA - c^T)x^* = 0 \text{ and } (y^*)^T(b - Ax^*) = 0.$$