École Polytechnique Fédérale de Lausanne
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Discrete Optimization (Spring 2025)

Assignment 5

1) Suppose you are given an oracle algorithm, which for a given polyhedron

$$P = \{\bar{x} \in \mathbb{R}^n : \bar{A}\bar{x} \leqslant \bar{b}\}$$

gives you a feasible solution or asserts that there is none. Show that using a single call of this oracle one can obtain an optimum solution for the LP

$$\max\{c^T x : x \in \mathbb{R}^n; Ax \leqslant b\}$$

assuming that the LP is feasible and bounded.

Solution:
The LP is feasible and bounded, thus an optimum solution must exist. Strong duality tells us that the dual $\min\{b^T y : A^T y = c, y \geqslant 0\}$ is feasible and bounded. For optimal solutions x^* of the primal and y^* of the dual we have

$$b^T y^* = c^T x^*.$$

Thus every point (x^*, y^*) of the polyhedron

$$\begin{aligned}
&c^T x = b^T y \\
&Ax \leqslant b \\
&A^T y = c \\
&y \geqslant 0
\end{aligned}$$

is optimal. Hence with one oracle call for the polyhedron above we get an optimal solution of the LP.

2) Determine the dual program for the following linear program:

$$\begin{aligned}
\min \quad & 3x_1 + 2x_2 - 3x_3 + 4x_4 \\
& 2x_1 - 2x_2 + 3x_3 + 4x_4 \leqslant 3 \\
& x2 + 3x3 + 4x4 \geqslant -5 \\
& 2x_1 - 3x_2 - 7x_3 - 4x_4 = 2 \\
& x_1 \geqslant 0 \\
& x_4 \leqslant 0
\end{aligned}$$

Solution:

$$\begin{aligned}
\max \quad & 3y_1 - 5y_2 + 2y_3 \\
& 2y_1 + 2y_3 \leqslant 3 \\
& -2y_1 + y_2 - 3y_3 = 2 \\
& 3y_1 + 3y_2 - 7y_3 = -3 \\
& y_1 + y_2 - y_3 \geqslant 1 \\
& y_1 \leqslant 0 \\
& y_2 \geqslant 0
\end{aligned}$$

3) Let $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ be a bounded, non-empty polyhedron. Formulate a linear program that computes the largest ball inside P.

Solution:
A ball of radius r and center x is contained in P if and only if $x \in P$ and x has distance at least r from any hyperplane defining P. Hence we obtain the following linear program:

$$\begin{aligned} \max \quad & r \\ \text{subject to} \quad & \frac{b_i - a_i x}{\|a_i\|} \geqslant r \quad \forall i = 1, \ldots, m \\ & Ax \leqslant b \end{aligned}$$

where $a_1, \ldots, a_m$ are the rows of A and $b = (b_1 \ldots b_m)^T$.

4) Consider the following linear program:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leqslant 6 \\ & x_1 + 2x_2 \leqslant 8 \\ & 3x_1 + 4x_2 \leqslant 22 \\ & x_1 + 5x_2 \leqslant 23 \end{aligned}$$

Show that $(4/3, 10/3)$ is an optimal solution by using weak duality.

Solution:
The assignment $(4/3, 10/3)$ has the objective function value of $14/3$. In order to prove that it is optimal (via weak duality), we are going to form the dual LP, and find a feasible solution to the dual that achieves the same objective value. The dual is:

$$\begin{aligned} \min \quad & 6y_1 + 8y_2 + 22y_3 + 23y_4 && (1) \\ \text{subject to} \quad & 2y_1 + y_2 + 3y_3 + y_4 = 1 && (2) \\ & y_1 + 2y_2 + 4y_3 + 5y_4 = 1 && (3) \\ & y1, y2, y3, y4 \geqslant 0 && (4) \end{aligned}$$

Thus, we are looking for a feasible dual solution such that $6y_1 + 8y_2 + 22y_3 + 23y_4 = 14/3$. By using Gaussian elimination on this constraint combined with (1) and (2) we get:

$$-4y_2 - 2y_3 - 7y_4 = -4/3$$

$$-3y_2 - 5y_3 - 9y_4 = -1$$

and

$$14/3y_3 + 5y_4 = 0.$$

Since $y_1, y_2, y_3, y_4 \geqslant 0$, we have that $y_3 = y_4 = 0$ and then $y_1 = y_2 = 1/3$. This is the desired feasible dual solution coinciding with the primal solution $(4/3, 10/3)$, proving the optimality of the latter.