École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

March 4, 2025

Discrete Optimization (Spring 2025)

Assignment 3

1) Using Theorem 3.11, prove the following variant of Farkas' lemma: Let $A \in \mathbb{R}^{m\times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax \leqslant b$, $x \in \mathbb{R}^n$ has a solution if and only if for all $\lambda \in \mathbb{R}^m_{\geqslant 0}$ with $\lambda^T A = 0$ one has $\lambda^T b \geqslant 0$.

2) Provide an example of a convex and closed set $K \subseteq \mathbb{R}^2$ and a linear objective function $c^T x$ such that $\inf\{c^T x \colon x \in K\} > -\infty$ but there does not exist an $x^* \in K$ with $c^T x^* \leqslant c^T x$ for all $x \in K$.

3) Consider the vectors

$$x_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, x_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

   The vector

$$v = x_1 + 3x_2 + 2x_3 + x_4 + 3x_5 = \begin{pmatrix} 15 \\ 14 \\ 25 \end{pmatrix}$$

   is a conic combination of the x_i.

   Write v as a conic combination using only three vectors of the x_i.

   Hint: Recall the proof of Carathéodory's theorem

4) In this exercise, assume that a linear program $\max\{c^T x \mid Ax \leqslant b\}$ can be solved in constant time $O(1)$. Suppose that $P(A, b)$ has vertices and that the linear program is bounded. Show how to compute an optimal *vertex* solution of the linear program in polynomial time in n and m where $A \in \mathbb{R}^{m\times n}$.

5) Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix and let $a_1, \dots, a_n \in \mathbb{R}^n$ be the columns of A. Show that $\operatorname{cone}(\{a_1, \dots, a_n\})$ is the polyhedron $P = \{y \in \mathbb{R}^n \colon A^{-1} y \geqslant 0\}$. Show that $\operatorname{cone}(\{a_1, \dots, a_k\})$ for $k \leqslant n$ is the set $P_k = \{y \in \mathbb{R}^n \colon a_i^{-1} x \geqslant 0, i = 1, \dots, k,\ a_i^{-1} x = 0, i = k+1, \dots, n\}$, where a_i^{-1} denotes the i-th row of A^{-1}.

6) Prove that for a finite set $X \subseteq \mathbb{R}^n$ the conic hull $\operatorname{cone}(X)$ is closed and convex.

   Hint: Use Carathéodory's theorem and exercise 5.