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March 4, 2025

Discrete Optimization (Spring 2025)

Assignment 3

1) Using Theorem 3.11, prove the following variant of Farkas' lemma: Let $A \in \mathbb{R}^{m\times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax \leqslant b$, $x \in \mathbb{R}^n$ has a solution if and only if for all $\lambda \in \mathbb{R}^m_{\geqslant 0}$ with $\lambda^T A = 0$ one has $\lambda^T b \geqslant 0$.

Solution:
We first show $Ax \leqslant b$ has a solution $\implies \forall \lambda \in \mathbb{R}^m_{\geqslant 0}$ with $\lambda^T A = 0, \lambda^T b \geqslant 0$.

Assume that there actually exists some $\lambda \in \mathbb{R}^m_{\geqslant 0}$ with $\lambda^T A = 0, \lambda^T b < 0$. Then

$$\begin{aligned}(\lambda^T A)x = 0^T x && = 0\\ \lambda^T(Ax) \leqslant \lambda^T b && < 0\end{aligned}$$

which is impossible as $\lambda^T Ax$ is distributive. Here we used in the inequality that $\lambda \geqslant 0$.

Now we show that $\forall \lambda \in \mathbb{R}^m_{\geqslant 0}$ with $\lambda^T A = 0$ we have $\lambda^T b \geqslant 0 \implies Ax \leqslant b$ has a solution.

Assume that $Ax \leqslant b$ actually has no solution. Now consider the matrix $\tilde{A} = \begin{pmatrix} A & -A & I_m \end{pmatrix}$ where the 3 matrices are concatenated and I_m is the $m \times m$ identity matrix. Then clearly $\tilde{A}x = b$ has no solution $x \geqslant 0$ as if there was such a solution where $x = (u, v, s)$ for some $u \in \mathbb{R}^n_{\geqslant 0}, v \in \mathbb{R}^n_{\geqslant 0}, s \in \mathbb{R}^m_{\geqslant 0}$, then we have that $Au - Av + s = b$ where $s \geqslant 0$ such that $A(u - v) \leqslant b$. This gives $(u - v) \in \mathbb{R}^n$ as a solution to $Ax \leqslant b$ which is assumed not to exist.

Thus $\tilde{A}x = b$ has no solution $x \geqslant 0$ such that the standard Farkas lemma tells us that there exists some $\lambda \in \mathbb{R}^m$ such that

$$\begin{aligned}\lambda^T b &< 0\\ \lambda^T \tilde{A} &\geqslant 0.\end{aligned}$$

Then

$$\begin{aligned}\lambda^T \tilde{A} &\geqslant 0\\ \implies \lambda^T \begin{pmatrix} A & -A & I_m \end{pmatrix} &\geqslant 0\\ \implies \lambda^T A \geqslant 0, -\lambda^T A \geqslant 0, \lambda &\geqslant 0\\ \implies \lambda^T A = 0, \lambda &\geqslant 0.\end{aligned}$$

Then λ satisfies $\lambda^T A = 0, \lambda^T b < 0, \lambda \geqslant 0$ which is a contradiction to the assumption. So $Ax \leqslant b$ must have a solution.

2) Provide an example of a convex and closed set $K \subseteq \mathbb{R}^2$ and a linear objective function $c^T x$ such that $\inf\{c^T x \colon x \in K\} > -\infty$ but there does not exist an $x^* \in K$ with $c^T x^* \leqslant c^T x$ for all $x \in K$.

Solution:
Let the set K be defined as follows:

$$K = \{x \in \mathbb{R}^2 : x_2 \geqslant e^{-x_1}\}.$$

Then clearly K is convex and closed by convexity of the function $f(x) = e^{-x}$. Now, let the linear objective $c^T x = x_2$. Then for any $x \in K, x_2 \geqslant e^{-x_1} \geqslant 0$ for any value $x_1 \in \mathbb{R}$ so that the objective $c^T x \geqslant 0 > -\infty$ for any $x \in K$.

Assume there exists some $x^* = (x_1^*, x_2^*) \in K$ such that $c^T x^* \leqslant c^T x$ for all $x \in K$. Take the point $\tilde{x} = (x_1^* + 1, e^{-(x_1^*+1)})$ which is also a point in K. Then

$$\tilde{x}_2 = e^{-(x_1^*+1)} < e^{-x_1^*} \leqslant x_2^*.$$

Then $c^T \tilde{x} < c^T x^*$ such that there does not exist such an x^*.

3) Consider the vectors

$$x_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, x_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, x_4 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, x_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The vector

$$v = x_1 + 3x_2 + 2x_3 + x_4 + 3x_5 = \begin{pmatrix} 15 \\ 14 \\ 25 \end{pmatrix}$$

is a conic combination of the x_i.

Write v as a conic combination using only three vectors of the x_i.

Hint: Recall the proof of Carathéodory's theorem

Solution:
Note that $-x_1 + x_3 + x_5 = \vec{0}$.

Let A be the matrix $\begin{pmatrix} | & | & | & | & | \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ | & | & | & | & | \end{pmatrix}$. Then

$$v = A \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \\ 3 \end{pmatrix} = A \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \\ 3 \end{pmatrix} + A \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = A \begin{pmatrix} 0 \\ 3 \\ 3 \\ 1 \\ 4 \end{pmatrix}.$$

Next, note that $x_3 + x_4 - 4x_5 = \vec{0}$. Then,

$$v = A \begin{pmatrix} 0 \\ 3 \\ 3 \\ 1 \\ 4 \end{pmatrix} = A \begin{pmatrix} 0 \\ 3 \\ 3 \\ 1 \\ 4 \end{pmatrix} + A \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -4 \end{pmatrix} = A \begin{pmatrix} 0 \\ 3 \\ 4 \\ 2 \\ 0 \end{pmatrix}$$

such that $v = 3x_2 + 4x_3 + 2x_4$.

4) In this exercise, assume that a linear program $\max\{c^T x \mid Ax \leqslant b\}$ can be solved in constant time $O(1)$. Suppose that $P(A, b)$ has vertices and that the linear program is bounded. Show how to compute an optimal *vertex* solution of the linear program in polynomial time in n and m where $A \in \mathbb{R}^{m \times n}$.

Solution:

Since $P(A, b)$ has vertices, we have $\operatorname{rank}(A) = n$. Recall Theorem 3.2 which shows that there is an optimal *vertex* solution of the linear program $\max\{c^T x \mid Ax \leqslant b\}$, if the linear program is feasible and bounded and $\operatorname{rank}(A) = n$. We will redo the proof of Theorem 3.2 in an algorithmic way.

First by using the "black box" constant time $O(1)$ algorithm, we get a feasible optimal solution x^*. Let $A_{x^*}x \leqslant b_{x^*}$ be the subsystem of $Ax \leqslant b$ that is satisfied by x^* with equalities. Define $\operatorname{rank}(x^*)$ to be the rank of A_{x^*}.

If x^* is a vertex, we are done. Otherwise, $\operatorname{rank}(x^*) < n$ and we will compute a feasible point $y^* \in P(A, b)$ such that

- $c^T y^* = c^T x^*$,
- $\operatorname{rank}(y^*) > \operatorname{rank}(x^*)$.

The procedure of computing y^* is as follows.

(a) Compute the matrix A_{x^*}, which can be done in polynomial time.

(b) Compute a non-zero kernel $d \in \mathbb{R}^n, d \neq 0$ of A_{x^*}, which can also be done in polynomial time.

(c) Compute the maximum distance $\lambda_{\max}$ to move along the same direction of d or the opposite direction of d, such that the updated point $y^* := x^* \pm \lambda_{\max} d$ is feasible, $c^T y^* = c^T x^*$, and $\operatorname{rank}(y^*) > \operatorname{rank}(x^*)$. Note that since x^* is optimal, we must have $c^T d = 0$ otherwise moving along d or $-d$ will strictly increase the objective value. Since $Ad \neq 0$ and $A_{x^*}d = 0$, we can compute an inequality of $Ax \leqslant b$, say $a_i^T x \leqslant b_i$, such that it's not in the subsystem $A_{x^*}x \leqslant b_{x^*}$ and $a_i^T d \neq 0$. Then take $\lambda_{\max} = \frac{b_i - a_i^T x^*}{|a_i^T d|}$, move along d if $a_i^T d > 0$ and move along $-d$ if $a_i^T d < 0$.

The procedure described above is in polynomial time of n, m. We keep doing the procedure at most n times until we get a feasible point whose rank is n, which is an optimal *vertex* solution.

5) Let $A \in \mathbb{R}^{n \times n}$ be a non-singular matrix and let $a_1, \ldots, a_n \in \mathbb{R}^n$ be the columns of A. Show that $\operatorname{cone}(\{a_1, \ldots, a_n\})$ is the polyhedron $P = \{y \in \mathbb{R}^n \colon A^{-1}y \geqslant 0\}$. Show that $\operatorname{cone}(\{a_1, \ldots, a_k\})$ for $k \leqslant n$ is the set $P_k = \{y \in \mathbb{R}^n \colon a_i^{-1}x \geqslant 0, i = 1, \ldots, k,\ a_i^{-1}x = 0, i = k+1, \ldots, n\}$, where a_i^{-1} denotes the i-th row of A^{-1}.

Solution:

Let $x \in \operatorname{cone}\{a_1, \ldots, a_k\}$. Then $x = \sum_{i=1}^{k} \lambda_i a_i$ such that $\lambda_i \geqslant 0$ for all i.

Then $x = \begin{pmatrix} a_1 & a_2 & \ldots & a_k \\ | & | & \ldots & | \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix}$.

Then $x = A \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ where $(\lambda, \vec{0})$ has $n-k$ zeros padded at the end. Then x satisfies $A^{-1}x = (\lambda, \vec{0})$ and since $\lambda_i \geqslant 0$ for all $i \in [k]$, $x \in P_k$. This shows $\text{cone}\{a_1, \dots, a_k\} \subseteq P_k$.

Next let $y \in P_k$ such that $a_i^{-1}y \geqslant 0$ for $i \in [k]$ and $a_i^{-1}y = 0$ for $i \in [k+1, n]$. Then $A^{-1}y = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ for some values $\lambda_i \geqslant 0, i \in [k]$ so that $y = \sum_{i=1}^{k} \lambda_i a_i + \sum_{i=k+1}^{n} 0 a_i$ so that $y \in \text{cone}\{a_1, \dots, a_k\}$. Thus $P_k \subseteq \text{cone}\{a_1, \dots, a_k\}$.

6) Prove that for a finite set $X \subseteq \mathbb{R}^n$ the conic hull $\text{cone}(X)$ is closed and convex.

Hint: Use Carathéodory's theorem and exercise 5.

Solution:

We show that $\text{cone}(X)$ is convex.

Let $z, y \in \text{cone}(X)$. Then $z = \sum_{x \in X} \lambda_x^z x, y = \sum_{x \in X} \lambda_x^y x$. Then $\lambda z + (1-\lambda)y = \sum_{x \in X} (\lambda \lambda_x^z + (1-\lambda)\lambda_x^y)x$ where $(\lambda \lambda_x^z + (1-\lambda)\lambda_x^y) \geqslant 0$ for every x as we are adding and multiplying non-negative values. Thus $\lambda z + (1-\lambda)y \in \text{cone}(X)$.

We show that $\text{cone}(X)$ is closed.

First, by Carathéodory's theorem, for each $y \in \text{cone}(X)$, there exists a linearly independent subset $\tilde{X} \subseteq X$ of size at most n, such that $y \in \text{cone}(\tilde{X})$. Therefore we have

$$\text{cone}(X) = \bigcup_{\substack{\tilde{X} \subseteq X, \\ |\tilde{X}| \leqslant n, \\ \tilde{X} \text{ is linearly independent}}} \text{cone}(\tilde{X}).$$

Since X is finite, the union above is also finite.

Next, we show that for every such $\text{cone}(\tilde{X})$ where $\tilde{X} \subseteq X, |\tilde{X}| \leqslant n$, and $\tilde{X}$ is linearly independent, $\text{cone}(\tilde{X})$ is closed. If $|\tilde{X}| = n$, then by Exercise 5, $\text{cone}(\tilde{X}) = \{y \in \mathbb{R}^n : A^{-1}y \geqslant 0\}$ where the columns of A are elements of $\tilde{X}$. Then for any convergent sequence (y_n) in $\text{cone}(\tilde{X})$ where $A^{-1}y_n \geqslant 0$ for each n. Let $y \in \mathbb{R}^n$ be the limit of (y_n). Since A^{-1} is continuous, $A^{-1}y_n \to A^{-1}y$, hence $A^{-1}y \geqslant 0$, i.e., $y \in \text{cone}(\tilde{X})$. If $|\tilde{X}| = k \leqslant n$, then we first extend (in whatever way) $\tilde{X}$ to get X' which is of size n and is linearly independent. Consider the matrix A where the columns of A are the elements of X'. Note that the first k columns of A are elements of $\tilde{X}$. By Exercise 5, $\text{cone}(\tilde{X}) = \{y \in \mathbb{R}^n : a_i^{-1}x \geqslant 0, i = 1, \dots, k \text{ and } a_i^{-1}x = 0, i = k+1, \dots, n\}$. For any convergent sequence (y_n) in $\text{cone}(\tilde{X})$, by a similar argument as before, one can show that the limit y of (y_n) satisfies $a_i^{-1}y \geqslant 0$ for $i = 1, \dots, k$ and $a_i^{-1}y = 0$ for $i = k+1, \dots, n$, i.e., $y \in \text{cone}(\tilde{X})$.

Since the finite union of closed sets is closed, this completes our proof.