École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

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Discrete Optimization (Spring 2025)

Assignment 2

1) Consider the unit ball $B_n = \{x \in \mathbb{R}^n \colon \|x\|_2 \leqslant 1\}$. Show that the set of extreme points of B is the sphere $S^{(n-1)} = \{x \in \mathbb{R}^n \colon \|x\|_2 = 1\}$.

2) A *line* is a set $L = \{x \cdot d + t \colon x \in \mathbb{R}\} \subseteq \mathbb{R}^n$ where $d, t \in \mathbb{R}^n$ $d \neq 0$. Show the following.

   A non-empty polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\} \subseteq \mathbb{R}^n$ contains a line if and only if $\operatorname{rank}(A) < n$.

3) Two different vertices $v_1 \neq v_2$ of a polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ are called *adjacent*, if there exists a subsystem $A'x \leqslant b'$ of $Ax \leqslant b$ with

   i) $A'v_1 = b'$ and $A'v_2 = b'$ and
   ii) $\operatorname{rank}(A') = (n-1)$.

   Show that there exists a valid inequality $c^T x \leqslant \delta$ of P with

$$\left(P \cap \{x \in \mathbb{R}^n \colon c^T x = \delta\}\right) = \operatorname{conv}\{v_1, v_2\}.$$

4) Let $\{C_i\}_{i \in I}$ be a family of convex subsets of $\mathbb{R}^n$. Show that the intersection $\bigcap_{i \in I} C_i$ is convex.

5) Show that the set of feasible solutions of a linear program is convex.

6) Let

$$P = \{x : Ax \leqslant b\}.$$

   Let $A^=$ denote the set of rows of A such that for all $x \in P$, $A^= x = b^=$ such that the rows indexed by $A^=$ are satisfied with equality in P. Prove that

$$\text{affine-hull}(P) = \{x \in \mathbb{R}^n : A^= x = b^=\} = \{x \in \mathbb{R}^n : A^= x \leqslant b^=\}.$$