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Discrete Optimization (Spring 2025)

Assignment 2

1) Consider the unit ball $B_n = \{x \in \mathbb{R}^n \colon \|x\|_2 \leqslant 1\}$. Show that the set of extreme points of B is the sphere $S^{(n-1)} = \{x \in \mathbb{R}^n \colon \|x\|_2 = 1\}$.

Solution:
First we prove that for any extreme point x^* of B_n, we have $\|x^*\|_2 = 1$. Let $\{x^*\} = B_n \cap \{\alpha^T x = \beta\}$. Suppose that $\|x^*\|_2 < 1$. Take $d \neq 0$ such that $\alpha^T d = 0$. For $\varepsilon > 0$, consider the point $\bar{x} = x^* + \varepsilon d$. Obviously we have $\alpha^T \bar{x} = \beta$. Since $\|x^*\|_2 < 1$, by taking ε to be small enough, we can make sure that $\|\bar{x}\|_2 \leqslant 1$, so $\bar{x} \in B_n$. Therefore $\bar{x} \neq x^*$ but $\bar{x} \in B_n \cap \{\alpha^T x = \beta\}$, a contradiction.

Next we prove that for any $x^* \in B_n$ such that $\|x^*\|_2 = 1$, x^* is an extreme point of B_n. Consider the hyperplane $(x^*)^T x = 1$, which is the tangent plane of S^{n-1} at x^*. Note that for any $x \in B_n$, we have $(x^*)^T x \leqslant 1$ so it is a supporting inequality of B_n. Also we have $\{x^*\} \subseteq B_n \cap \{(x^*)^T x = 1\}$. For any $y \in B_n \cap \{(x^*)^T x = 1\}$, let $\theta \in [0, \pi)$ be the angle between x^* and y, we have

$$1 = (x^*)^T y = \|x^*\|_2 \|y\|_2 \cos\theta \leqslant 1 \Rightarrow \|y\|_2 = 1, \theta = 0 \Rightarrow y = x^*.$$

Therefore $\{x^*\} = B_n \cap \{(x^*)^T x = 1\}$, so x^* is an extreme point of B_n.

2) A *line* is a set $L = \{x \cdot d + t \colon x \in \mathbb{R}\} \subseteq \mathbb{R}^n$ where $d, t \in \mathbb{R}^n$ $d \neq 0$. Show the following.

A non-empty polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\} \subseteq \mathbb{R}^n$ contains a line if and only if $\text{rank}(A) < n$.

Solution:
We start by showing P contains a line $\implies$ $\text{rank}(A) < n$.

Let $d \cdot x + t$ denote the line contained in P. As P contains this line, for any $x \in \mathbb{R}$, $A(d \cdot x + t) \leqslant b$. We claim that $Ad = \vec{0}$ in this case. Assume that $Ad \neq 0$ and let i be any component of Ad that is nonzero. Let the value of this component be equal to $\alpha \in \mathbb{R}_{>0}$ without loss of generality (an identical argument would apply for α negative). Then

$$\begin{aligned} [A(d \cdot x + t)]_i &= x[Ad]_i + [At]_i \\ &= x\alpha + [At]_i \end{aligned}$$

for any choice of $x \in \mathbb{R}$. In particular, choose $x := \frac{b_i - [At]_i + 1}{\alpha}$ which is possible since $\alpha \neq 0$. Then

$$\begin{aligned} [A(d \cdot x + t)]_i &= x[Ad]_i + [At]_i \\ &= x\alpha + [At]_i \\ &= b_i - [At]_i + 1 + [At]_i \\ &> b_i. \end{aligned}$$

But this contradicts the fact that $A(d \cdot x + t) \leqslant b$ and in particular $[A(d \cdot x + t)]_i \leqslant b_i$ for any $x \in \mathbb{R}$. Thus it must be that $Ad = 0$ such that d is a nontrivial kernel element of A and $\text{rank}(A) < n$.

We then show that if $\text{rank}(A) < n$ then P contains a line.

As $\text{rank}(A) < n$, there exists a non-trivial kernel element of A, a vector $v \neq \vec{0}$ such that $Av = 0$. Then let $\vec{x}$ be any feasible vector of the polyhedron P, that is x is such that $Ax \leqslant b$. Note that for any $t \in \mathbb{R}$, $A(x + t \cdot v) = Ax + tAv = Ax + \vec{\{0\}} \leqslant b$ such that the vector $x + t \cdot v$ is contained in P for any $t \in \mathbb{R}$. This forms the line contained in P.

3) Two different vertices $v_1 \neq v_2$ of a polyhedron $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ are called *adjacent*, if there exists a subsystem $A'x \leqslant b'$ of $Ax \leqslant b$ with

   i) $A'v_1 = b'$ and $A'v_2 = b'$ and
   ii) $\text{rank}(A') = (n-1)$.

   Show that there exists a valid inequality $c^T x \leqslant \delta$ of P with

$$\big(P \cap \{x \in \mathbb{R}^n \colon c^T x = \delta\}\big) = \text{conv}\{v_1, v_2\}.$$

Solution:
Let A' be the rank $n-1$ submatrix such that $A'v_1 = b', A'v_2 = b'$. Let $c^T = \vec{1}^T A'$ be the vector obtained by summing the rows of A' and let $\delta = \vec{1}^T b'$ the value obtained by summing the components of b'. As A' can be chosen as $(n-1)$ linearly independent rows, the vector $\vec{c}$ is nonzero.

Now, we claim that

$$(P \cap \{x \in \mathbb{R}^n : c^T x = \delta\}) = \text{conv}\{v_1, v_2\}.$$

Let x be any convex combination of v_1, v_2 such that $x = \lambda v_1 + (1-\lambda) v_2$. Notice that $x \in P$ as $v_1, v_2 \in P$ and P is convex. Next, note that

$$\begin{aligned} c^T x &= c^T(\lambda v_1 + (1-\lambda) v_2) \\ &= (\vec{1}^T A')(\lambda v_1 + (1-\lambda) v_2) \\ &= \lambda(\vec{1}^T A' v_1) + (1-\lambda)(\vec{1}^T A' v_2) \\ &= \lambda(\vec{1}^T b') + (1-\lambda)(\vec{1}^T b') \\ &= \lambda\delta + (1-\lambda)\delta \\ &= \delta. \end{aligned}$$

This shows that $x \in (P \cap \{x \in \mathbb{R}^n : c^T x = \delta\})$ so that $\text{conv}\{v_1, v_2\} \subseteq (P \cap \{x \in \mathbb{R}^n : c^T x = \delta\})$.

Next, let $x \in (P \cap \{x \in \mathbb{R}^n : c^T x = \delta\})$ any such vector, and we show that $x \in \text{conv}\{v_1, v_2\}$.

As x satisfies $c^T x = \delta$, we claim that $A'x = b'$. Assume this were not the case such that there is a row i with $A'_i x < b'_i$. Then $c^T x = \vec{1}^T A' x < \vec{1}^T b' = \delta$. But this is impossible as $c^T x = \delta$. Thus $A'x = b'$. As v_1, v_2 are vertices, there exists some row $a^{(1)}$ of A such that $\begin{pmatrix} A' \\ a^{(1)} \end{pmatrix} v_1 = \begin{pmatrix} b' \\ b^{(1)} \end{pmatrix}$ and $\text{rank} \begin{pmatrix} A' \\ a^{(1)} \end{pmatrix} = n$. Let the matrix $\begin{pmatrix} A' \\ a^{(1)} \end{pmatrix}$ be denoted as $A^{(1)}$.

Then note that $A^{(1)} x = \begin{pmatrix} b' \\ \alpha_x \end{pmatrix}$ for some $\alpha_x \leqslant b^{(1)}$. Likewise, $A^{(1)} v_2 = \begin{pmatrix} b' \\ \alpha_{v_2} \end{pmatrix}$ for $\alpha_{v_2} \leqslant b^{(1)}$.

Assume that $\alpha_{v_2} \leqslant \alpha_x$. Then letting $\lambda = \frac{\alpha_x - \alpha_{v_2}}{b^{(1)} - \alpha_{v_2}}$ where $\lambda \in [0,1]$ as $\alpha_{v_2} \leqslant \alpha_x \leqslant b^{(1)}$. Then

$$\begin{aligned}
\lambda v_1 + (1-\lambda) v_2 &= \lambda A^{(1)^{-1}} \begin{pmatrix} b' \\ b^{(1)} \end{pmatrix} + (1-\lambda) A^{(1)^{-1}} \begin{pmatrix} b' \\ \alpha_{v_2} \end{pmatrix} \\
&= A^{(1)^{-1}} \cdot \begin{pmatrix} b' \\ \lambda b^{(1)} + (1-\lambda)\alpha_{v_2} \end{pmatrix} \\
&= A^{(1)^{-1}} \cdot \begin{pmatrix} b' \\ \frac{\alpha_x - \alpha_{v_2}}{b^{(1)} - \alpha_{v_2}} b^{(1)} + \left(1 - \frac{\alpha_x - \alpha_{v_2}}{b^{(1)} - \alpha_{v_2}}\right) \alpha_{v_2} \end{pmatrix} \\
&= A^{(1)^{-1}} \cdot \begin{pmatrix} b' \\ \alpha_x \end{pmatrix} \\
&= x
\end{aligned}$$

so that x is indeed a convex combination of v_1 and v_2. Finally, note that if instead $\alpha_x \leqslant \alpha_{v_2}$ then v_2 is a convex combination of x and v_1 which is impossible as v_2 is a vertex.

So we have shown that $(P \cap \{x \in \mathbb{R}^n : c^T x = \delta\}) \subseteq \operatorname{conv}\{v_1, v_2\}$ such that altogether

$$(P \cap \{x \in \mathbb{R}^n : c^T x = \delta\}) = \operatorname{conv}\{v_1, v_2\}.$$

4) Let $\{C_i\}_{i \in I}$ be a family of convex subsets of $\mathbb{R}^n$. Show that the intersection $\bigcap_{i \in I} C_i$ is convex.

Solution:
Let x, y be two vectors in the set $\bigcap_{i \in I} C_i$. Then $x, y \in C_i$ for each $i \in I$. Then $\lambda x + (1-\lambda) y \in C_i$ for each $i \in I$ by convexity of C_i. But this shows that $\lambda x + (1-\lambda) y \in \bigcap_{i \in I} C_i$ such that the intersection is also convex.

5) Show that the set of feasible solutions of a linear program is convex.

Solution:
Let the feasible region of the LP be defined by some $\{x : Ax \leqslant b\}$ for a matrix A and vector b. Let x, y be any two feasible solutions of the LP. Then

$$\begin{aligned}
A(\lambda x + (1-\lambda) y) &= \lambda A x + (1-\lambda) A y \\
&\leqslant \lambda b + (1-\lambda) b \\
&= b
\end{aligned}$$

such that the convex combination is also in the feasible region.

6) Let

$$P = \{x : Ax \leqslant b\}.$$

Let $A^=$ denote the set of rows of A such that for all $x \in P$, $A^= x = b^=$ such that the rows indexed by $A^=$ are satisfied with equality in P. Prove that

$$\text{affine-hull}(P) = \{x \in \mathbb{R}^n : A^= x = b^=\} = \{x \in \mathbb{R}^n : A^= x \leqslant b^=\}.$$

Solution:
We prove this by proving three containments.

(a) affine-hull$(P) \subseteq \{x : A^{=}x = b^{=}\}$.

By definition we have that $P \subseteq \{x : A^{=}x = b^{=}\}$. Let $x \in$ affine-hull(P) be any vector. Then $x = \lambda_1 x^1 + \ldots + \lambda_t x^t$ for some $x^1, \ldots, x^t \in P, \lambda_1, ..., \lambda_t \in \mathbb{R}, \sum_{i=1}^{t} \lambda_i = 1$. Then

$$\begin{aligned} A^{=}x &= \lambda_1 A^{=}x^1 + \ldots \lambda_t A^{=}x^t \\ &= \sum_{i=1}^{t} \lambda_i b^{=} \\ &= b^{=}. \end{aligned}$$

Thus $x \in \{x : A^{=}x = b^{=}\}$ and the containment follows.

(b) $\{x : A^{=}x = b^{=}\} \subseteq \{x : A^{=}x \leqslant b^{=}\}$. This containment is immediate by the definition of $A^{=}$.

(c) $\{x : A^{=}x \leqslant b^{=}\} \subseteq$ affine-hull(P).

Let x be a vector satisfying $A^{=}x \leqslant b^{=}$. Denote the submatrix of rows not in $A^{=}$ by $A^{<}$. We claim that there exists a point $x' \in P$ such that $A^{=}x' = b^{=}$ and $A^{<}x' < b^{<}$ (all the inequalities are strict). To prove the claim, denote all the inequalitiess of $A^{<}x \leqslant b^{<}$ as $a_1^T x \leqslant b_1, \ldots, a_k^T x \leqslant b_k$. Then for each $i = 1, \ldots, k$, there exists a point $x_i \in P$ such that $a_i^T x_i < b_i$ by definition of $A^{=}$. Take $x' = \frac{1}{k} \sum_{i=1}^{k} x_i$. This finishes the proof of the claim.

If $x \in P$ then $x \in$ affine-hull(P), we are done. Otherwise if $x \notin P$, consider $x'' = x' + \varepsilon(x - x')$ where $\varepsilon \geqslant 0$. First it's easy to check that $A^{=}x'' \leqslant b^{=}$. Also by taking ε to be small enough, we can make sure that $A^{<}x'' \leqslant b^{<}$. Therefore $x'' \in P$. Consider the line $L =$ affine-hull$(\{x', x''\})$ such that

$$\text{affine-hull}(P) \supseteq \text{affine-hull}(\{x', x''\}) \ni x$$

which completes the proof.