École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

February 18, 2025

Discrete Optimization (Spring 2025)

Assignment 1

Problem 1

Provide a certificate (as in Theorem 1.1 in the lecture notes) of the unsolvability of the linear equation

$$\begin{pmatrix} 2 & 1 & 0 \\ 5 & 4 & 1 \\ 7 & 5 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

Problem 2

Let $A \in \mathbb{R}^{3\times 2}$ be the matrix

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 0 & -1 \end{pmatrix}$$

and $b \in \mathbb{R}^3$ be the vector

$$b = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

defining the system of inequalities $Ax \leq b$ that does not have a feasible solution. Find a Farkas' certificate, i.e., a $\lambda \in \mathbb{R}^3_{\geq 0}$ with $\lambda^T A = 0$ and $\lambda^T b = -1$.

Problem 3

Show the "if" direction of the Farkas' lemma: given $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$, if there exist a $\lambda \in \mathbb{R}^m_{\geqslant 0}$ such that $\lambda^\top A = 0$ and $\lambda^\top b = -1$, then the system $Ax \leq b$ of linear inequalities does not have a solution.

Problem 4

Consider the following linear program:

$$\begin{array}{rrcrcr} \max & x & + & y & & \\ \text{s.t.} & 3x & + & 2y & \leq & 6 \\ & x & + & 4y & \leq & 4. \end{array}$$

The solution $(x, y) = (8/5, 3/5)$ satisfies the both constraints and has the objective value $11/5$. Provide a certificate that this is an optimal solution.

Problem 5

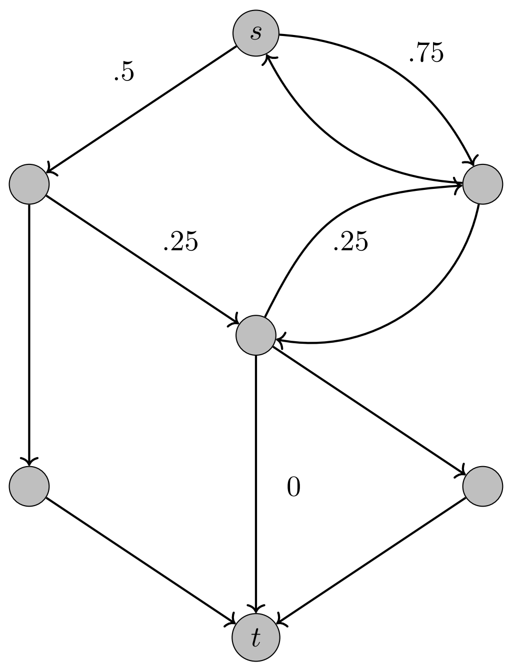
Let $G = (V, A)$ be a directed graph and $s, t \in V$ be two designated vertices. For a vertex $v \in V$ we let

$$\delta^+(v) = \{(u, v) \colon u \in V,\ (u, v) \in A\} \text{ and } \delta^-(v) = \{(v, u) \colon u \in V,\ (v, u) \in A\}$$

the *arcs entering* and *leaving* v respectively. Consider the following inequalities

$$\begin{aligned}
\sum_{a \in \delta^+(v)} x_a - \sum_{a \in \delta^-(v)} x_a &= 0 \qquad v \in V \setminus \{s, t\} \\
\sum_{a \in \delta^+(s)} x_a - \sum_{a \in \delta^-(s)} x_a &= -1 \\
\sum_{a \in \delta^+(t)} x_a - \sum_{a \in \delta^-(t)} x_a &= 1 \\
x_a &\geq 0 \qquad a \in A.
\end{aligned} \tag{1}$$

a) Consider the following digraph with s and t and a partial assignment of arc variables. Can this partial assignment be completed to a feasible solution satisfying the inequalities (1)? If yes, complete the assignment.



b) Show the following for a digraph $G = (V, A)$ with $s, t \in V$: If there is a path connecting s and t in G, then the system of inequalities (1) has a feasible solution

c) (*) Show the following for a digraph $G = (V, A)$ with $s, t \in V$: If the system of inequalities (1) has a feasible solution, then there is a path connecting s and t in G.