École Polytechnique Fédérale de Lausanne
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Discrete Optimization (Spring 2025)

Assignment 1

Problem 1

Provide a certificate (as in Theorem 1.1 in the lecture notes) of the unsolvability of the linear equation

$$\begin{pmatrix} 2 & 1 & 0 \\ 5 & 4 & 1 \\ 7 & 5 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

Solution: Consider $y = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ so that $y^T A = \begin{pmatrix} 7-7 \\ 5-5 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $y^T b = 3 - 4 = -1$.

Then $y^T A = \vec{0}$ and $y^T b < 0$ such that the system must be infeasible. See that if there did exist a solution $\vec{x}$, then

$$\begin{aligned} y^T Ax &= y^T b \\ \implies \vec{0}^T x &= y^T b \\ \implies 0 &= -1. \end{aligned}$$

As this is impossible, the system must be infeasible.

Problem 2

Let $A \in \mathbb{R}^{3\times 2}$ be the matrix

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 0 & -1 \end{pmatrix}$$

and $b \in \mathbb{R}^3$ be the vector

$$b = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

defining the system of inequalities $Ax \leq b$ that does not have a feasible solution. Find a Farkas' certificate, i.e., a $\lambda \in \mathbb{R}^3_{\geq 0}$ with $\lambda^T A = 0$ and $\lambda^T b = -1$.

Solution: Let $\lambda = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix}$. Then $\lambda^T A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ while $\lambda^T b = -1$ such that this is a certificate of infeasibility.

Problem 3

Show the "if" direction of the Farkas' lemma: given $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m$, if there exist a $\lambda \in \mathbb{R}^m_{\geqslant 0}$ such that $\lambda^\top A = 0$ and $\lambda^\top b = -1$, then the system $Ax \leq b$ of linear inequalities does not have a solution.

Solution: Suppose there exists a vector $\vec{x}$ such that $Ax \leq b$. Then we have that

$$\begin{aligned}\lambda^T Ax &= \lambda^T(Ax) \\ &\leq \lambda^T b \\ &= -1\end{aligned}$$

where the inequality follows from $\lambda \geq \vec{0}$.
But

$$\begin{aligned}\lambda^T Ax &= (\lambda^T A)x \\ &= \vec{0}^T x \\ &= 0\end{aligned}$$

so that with the two equations together, we get $0 \leq -1$ which is a contradiction.

Problem 4

Consider the following linear program:

$$\begin{array}{lcccccc} \max & x & + & y & & \\ \text{s.t.} & 3x & + & 2y & \leq & 6 \\ & x & + & 4y & \leq & 4. \end{array}$$

The solution $(x, y) = (8/5, 3/5)$ satisfies the both constraints and has the objective value $11/5$. Provide a certificate that this is an optimal solution.

Solution: The LP for our problem is given by

$$\begin{aligned}&\max(x+y) \\ s.t. &\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 6 \\ 4 \end{pmatrix}.\end{aligned}$$

Consider the following linear combination of the two constraints given by

$$\lambda(3x+2y) + \alpha(x+4y)$$

where $\lambda, \alpha \geq 0$. Clearly, this preserves the inequalities so that

$$\lambda(3x+2y) + \alpha(x+4y) \leq 6\lambda + 4\alpha$$

for any feasible x, y for the LP.
Now, we choose $\lambda = 3/10, \alpha = 1/10$ and get that for any feasible x, y for the LP,

$$\begin{aligned}&\lambda(3x+2y) + \alpha(x+4y) \leq 6\lambda + 4\alpha \\ \Longrightarrow\ &9/10x + 6/10y + 1/10x + 4/10y \leq 18/10 + 4/10 \\ \Longrightarrow\ &x + y \leq 22/10 \\ \Longrightarrow\ &x + y \leq 11/5.\end{aligned}$$

This shows that any feasible solution satisfies the objective $x+y \leq 11/5$ such that $(x, y) = (8/5, 3/5)$ satisfies the both constraints and has the optimal objective value $11/5$.

Problem 5

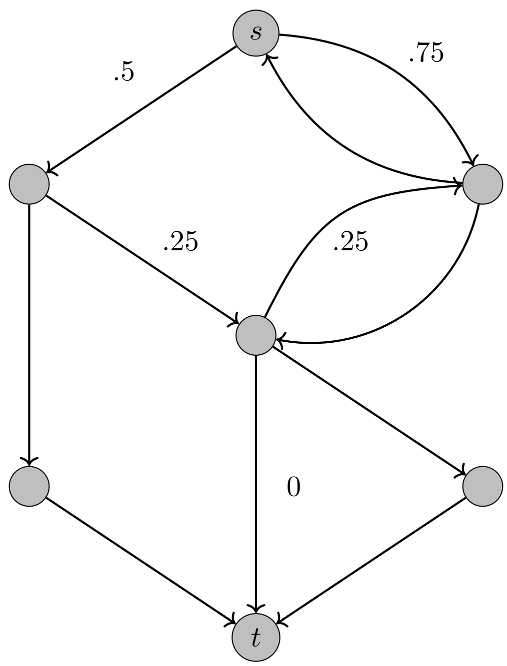
Let $G = (V, A)$ be a directed graph and $s, t \in V$ be two designated vertices. For a vertex $v \in V$ we let

$$\delta^+(v) = \{(u, v)\colon u \in V,\ (u, v) \in A\} \text{ and } \delta^-(v) = \{(v, u)\colon u \in V,\ (v, u) \in A\}$$

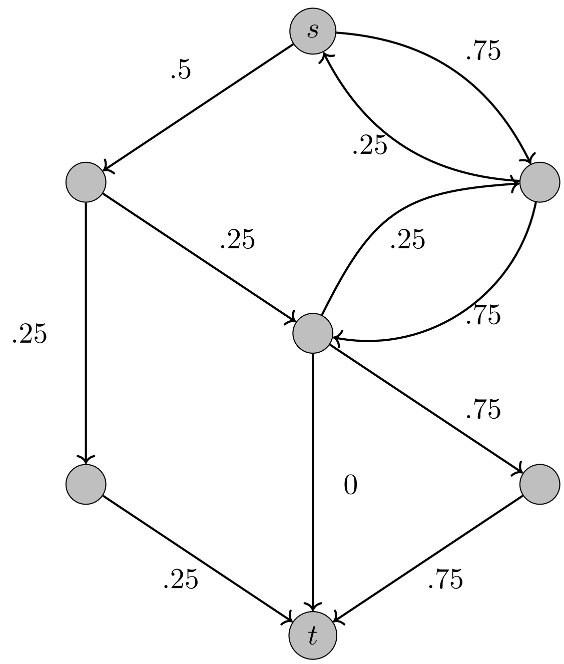
the *arcs entering* and *leaving* v respectively. Consider the following inequalities

$$\begin{aligned} \sum_{a \in \delta^+(v)} x_a - \sum_{a \in \delta^-(v)} x_a &= 0 \qquad v \in V \setminus \{s, t\} \\ \sum_{a \in \delta^+(s)} x_a - \sum_{a \in \delta^-(s)} x_a &= -1 \\ \sum_{a \in \delta^+(t)} x_a - \sum_{a \in \delta^-(t)} x_a &= 1 \\ x_a &\geq 0 \qquad a \in A. \end{aligned} \tag{1}$$

a) Consider the following digraph with s and t and a partial assignment of arc variables. Can this partial assignment be completed to a feasible solution satisfying the inequalities (1)? If yes, complete the assignment.



Solution:



b) Show the following for a digraph $G = (V, A)$ with $s, t \in V$: If there is a path connecting s and t in G, then the system of inequalities (1) has a feasible solution

Solution: Given a path t to s, set for every edge on the path the value $x_e = +1$ and for every edge not on the path the flow value $x_e = 0$. Given this choice of flow, it is not hard to see that all constraints are satisfied by this solution.

c) (*) Show the following for a digraph $G = (V, A)$ with $s, t \in V$: If the system of inequalities (1) has a feasible solution, then there is a path connecting s and t in G.

Solution: Suppose that the system of inequalities has a feasible solution $\vec{x}$ but that there is no path from s to t. Call A the set of nodes that are reachable from s using edges (u, v) that have $x(u, v) > 0$. Then A contains s and it does not contain t as we assume that t is not reachable from s. Thus the set $A \subset V$ is a cut where A contains s and $V \setminus A$ contains t. The net value out of A is equal to $\sum_{(u,v):u \in A, v \notin A} x_{(u,v)}$. However by definition of A, any such edge that exits A must have value 0 in the vector x as otherwise the node would be reachable from s using positive valued edges. This means that for any $v \in V \setminus A$, $\sum_{a \in \delta^+(v)} x_a = 0$. In order to satisfy the first equality, this means that for every $v \in V \setminus A - t$, it must also be that $\sum_{a \in \delta^-(v)} x_a = 0$. But then t receives 0 value on incoming edges as it is only reachable by nodes in $V \setminus A$ such that the third equality of the LP is not satisfied. This is a contradiction to $\vec{x}$ being feasible.

Alternative solution: Suppose that the system of inequalities has a feasible solution $\vec{x}$ but that there is no path from s to t. Let S be the set of vertices that are reachable by a path from s. By our assumption S contains s and does not contain t. Then by summing up the equalities in (1) for $v \in S$ we get

$$
\begin{aligned}
&\sum_{v\in S}\left(\sum_{a\in\delta^+(v)} x_a - \sum_{a\in\delta^-(v)} x_a\right)\\
&= \left(\sum_{a\in\delta^+(s)} x_a - \sum_{a\in\delta^-(s)} x_a\right) + \sum_{v\in S\setminus\{s\}}\left(\sum_{a\in\delta^+(v)} x_a - \sum_{a\in\delta^-(v)} x_a\right)\\
&= -1 + 0 = -1,
\end{aligned}
$$

which implies that there must be an arc $a = (u, v), u \in S, v \notin S$ with $x_a > 0$. Since u is reachable by a path P_u from s, v is also reachable by path $P_u \cup \{a\}$, contradiction to $v \notin S$.