École Polytechnique Fédérale de Lausanne
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Discrete Optimization (Spring 2025)

Assignment 13

1) Let $A \in \mathbb{Z}^{m\times n}$ be a matrix of rank n and let $b \in \mathbb{Z}^m$. Show the following: If $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ is full-dimensional, then there exist $n+1$ vertices of P that are *affinely* independent.

2) Given vectors $c \in \mathbb{R}^n$ and $a \in \mathbb{R}^n$, and a symmetric positive definite matrix $A \in \mathbb{R}^{n\times n}$, provide a formula for the ellipsoid containing the half-ball $H = \{x \in \mathbb{R}^n \colon \| x \|_2 \leqslant 1, c^T x \geqslant 0\}$.

3) Let $n \in \mathbb{N}$ and consider the space $\mathbb{R}^{n^2}$. An element $x \in \mathbb{R}^{n^2}$ be interpreted as a matrix $A \in \mathbb{R}^{n\times n}$ in the obvious way as

$$A = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_{n+1} & x_{n+2} & \cdots & x_{2n} \\ & & \vdots & \\ x_{n^2-n+1} & x_{n^2-n+2} & \cdots & x_{n^2} \end{pmatrix}$$

Let $X \subseteq \mathbb{R}^{n^2}$ be the subset of $\mathbb{R}^{n^2}$ consisting of symmetric and positive semidefinite matrices.

   i) Show that X is convex.

   ii) Let $A \in \mathbb{R}^{n\times n}$, $A \notin X$. Describe a hyperplane $a^T x = \beta$ that separates A from X.

4) Give an example for a linear program with no maximum (in other words, unbounded linear program) such that the corresponding integer program is not unbounded.

5) Let $E \subset \mathbb{R}^3$ be the ellipsoid $E = \{(x,y,z) : x^2 + \frac{y^2}{4} + \frac{z^2}{9} \leqslant 1\}$. Let H^+ be the half-space $H^+ = \{(x,y,z) : x+y+z \geqslant 0\}$. Find an ellipsoid E' such that $E' \supset E \cap H^+$ and $\mathrm{Vol}(E') \leqslant \mathrm{Vol}(E) \cdot e^{-1/(2(3+1))}$.

6) (Bonus Question) Suppose we are given an oracle that tells us whether a polyhedron defined by $Ax \leqslant b$ is full-dimensional. Based on this oracle, this exercise develops a method to find an inequality $a^T x \leqslant \beta$ of $Ax \leqslant b$ that is satisfied by every feasible solution with *equality* in the case where $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ is not full-dimensional.

   First we split $Ax \leqslant b$ into two systems $A_1 x \leqslant b_1$ and $A_2 x \leqslant b_2$, where $A_1 x \leqslant b_1$ are the inequalities that are satisfied with equality by every feasible $x^* \in \mathbb{R}^n$. The system $A_1 x \leqslant b_1$ is called the *implicit equalities* of $Ax \leqslant b$.

   i) If $Ax \leqslant b$ is feasible, then there exists a feasible solution x^* such that $A_2 x^* < b_2$ holds.

   ii) Argue that the implicit equalities of $A_1 x \leqslant b_1$ are $A_1 x \leqslant b_1$.

   iii) Suppose that $Ax \leqslant b$ is not full-dimensional and that $A'x \leqslant b'$ is full-dimensional, where $A'x \leqslant b'$ stems from $Ax \leqslant b$ by deleting one inequality $a^T x \leqslant \beta$. Show that $a^T x \leqslant \beta$ is an implicit equality of $Ax \leqslant b$.

   iv) If $a^T x \leqslant \beta$ is an implicit equality of $Ax \leqslant b$, then describe a feasibility problem in $\mathbb{R}^{n-1}$ that is equivalent to the one of $Ax \leqslant b$.