Discrete Optimization (Spring 2025)

Assignment 13

1) Let $A \in \mathbb{Z}^{m \times n}$ be a matrix of rank n and let $b \in \mathbb{Z}^m$. Show the following: If $P = \{x \in \mathbb{R}^n \colon Ax \leqslant b\}$ is full-dimensional, then there exist $n + 1$ vertices of P that are *affinely* independent.

   Solution:
   Note that P is a polytope and thus the convex hull of its vertices. We will consecutively choose vertices of P while ensuring that each newly selected vertex is affinely independent to the previous ones. Initially, we can choose the first vertex arbitrarily, as every single point is affinely independent. Now, assume that we have already chosen vertices $v_1, ..., v_k$. Out of the remaining vertices we pick as v_{k+1} an arbitrary vertex of P such that $v_1, ..., v_k, v_{k+1}$ is affinely independent. We continue this process until k is such that all the remaining vertices are affinely dependent with $v_1, ..., v_k$. This means that all the vertices and thus their convex hull P are contained in the $k - 1$-dimensional affine subspace defined by $v_1, ..., v_k$. Since P is full-dimensional, this can only happen for $k = n + 1$. At that point we are done.

2) Given vectors $c \in \mathbb{R}^n$ and $a \in \mathbb{R}^n$, and a symmetric positive definite matrix $A \in \mathbb{R}^{n \times n}$, provide a formula for the ellipsoid containing the half-ball $H = \{x \in \mathbb{R}^n \colon \| x \|_2 \leqslant 1, c^T x \geqslant 0\}$.

   Solution:
   For convenience, we use a different representation of the ellipsoid than before. Here, we define it as

   $$E(A, a) = \{x \in \mathbb{R}^n : (x - a)^T A^{-1}(x - a) \leqslant 1\}$$

   where $A \in \mathbb{R}^{n \times n}$ is a positive definite (PD) matrix, and $a \in \mathbb{R}^n$ is the center. We remark that for every PD matrix A there exists a unique PD matrix, denote it with $A^{1/2}$, such that $A = A^{1/2}A^{1/2}$. Thus, $E(A, a) = \{x \in \mathbb{R}^n : (x - a)^T (A^{1/2})^{-1}(A^{1/2})^{-1}(x - a) \leqslant 1\}$.

   Denote with $e_1 \in \mathbb{R}^n$ the unit vector with the first coordinate equal to 1, and with I_n the $n \times n$ identity matrix. When rewriting the ellipsoid, we obtain that the half-ball $\{x \in \mathbb{R}^n \colon \| x \| \leqslant 1, e^T 1x \geqslant 0\}$ is contained in the ellipsoid $E(A(e_1), a(e_1))$, where

   $$A(e1) = \frac{n^2}{n^2 - 1}\left(I_n - \frac{2}{n + 1}e_1 e_1^T\right) \quad \text{and } a(e_1) = \frac{1}{n + 1}e_1.$$

   Without loss of generality we could assume that $\| c \| = 1$. Similarly to the proof of the Half-ball lemma in the lecture notes, one can verify that e_1 can be replaced with c in the above formulas. I.e., the ellipsoid $E(A(c), a(c))$ contains the half-ball H given in the problem statement.

3) Let $n \in \mathbb{N}$ and consider the space $\mathbb{R}^{n^2}$. An element $x \in \mathbb{R}^{n^2}$ be interpreted as a matrix $A \in \mathbb{R}^{n \times n}$ in the obvious way as

   $$A = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_{n+1} & x_{n+2} & \cdots & x_{2n} \\ & & \vdots & \\ x_{n^2-n+1} & x_{n^2-n+2} & \cdots & x_{n^2} \end{pmatrix}$$

   Let $X \subseteq \mathbb{R}^{n^2}$ be the subset of $\mathbb{R}^{n^2}$ consisting of symmetric and positive semidefinite matrices.

i) Show that X is convex.

ii) Let $A \in \mathbb{R}^{n \times n}$, $A \notin X$. Describe a hyperplane $a^T x = \beta$ that separates A from X.

Solution:

i) Let $x, y \in X$ and let their matrix representations be A_x, A_y. Then for any $\lambda \in [0, 1]$, take $z = \lambda x + (1 - \lambda)y$. Then $A_z = \lambda A_x + (1 - \lambda)A_y$ and for any indeces i, j,

$$A_z[ij] = \lambda A_x[ij] + (1 - \lambda)A_y[ij] = \lambda A_x[ji] + (1 - \lambda)A_y[ji] = A_z[ji]$$

where the second equality follows from symmetry of A_x and A_y. Hence, A_z is symmetric. Moreover, consider any vector $v \in \mathbb{R}^n$, then

$$v^T A_z v = v^T(\lambda A_x + (1 - \lambda)A_y)v = \lambda v^T A_x v + (1 - \lambda)v_y^A v \geqslant 0 + 0 = 0$$

where the inequality follows from A_x, A_y being PSD. Then A_z is also PSD and therefore z is in X. Thus X is convex.

ii) Let a be the vector representation of A. Since $A \notin X$, A is either not symmetric or not PSD. In the case where A is not symmetric, there exists indeces i, j such that $A[ij] > A[ji]$ without loss of generality. Notice that in a, $A[ij]$ corresponds to the component $a_{(i-1) \cdot n + j}$. Then letting the vector α be defined by

$$\alpha_k = \begin{cases} 0 & \text{if } k \notin \{(i-1) \cdot n + j, (j-1) \cdot n + i\} \\ 1 & \text{if } k = (i-1) \cdot n + j \\ -1 & \text{if } k = (j-1) \cdot n + i \end{cases}.$$

Then $\alpha^T a > 0$ by the assumption $A[ij] > A[ji]$, but $\alpha^T x = 0$ for all $x \in X$ as these vectors are symmetric. Thus $\alpha^T x = 0$ is a separating hyperplane.

Now consider the case where A is not PSD. Then there exists some vector $v \in \mathbb{R}^n$ such that $v^T A v < 0$. Note that $v^T A v = \sum_{i=1}^{n} \sum_{j=1}^{n} v_i A[ij]v_j$. Recall that $A[ij] = a_{(i-1) \cdot n + j}$. Then let α be the vector where $\alpha_{(i-1) \cdot n + j} = v_i v_j$ for each $i \in [n], j \in [n]$. Notice then that $\alpha \in \mathbb{R}^{n^2}$ and $\alpha^T a = v^T A v < 0$. Moreover, for any $x \in X, \alpha^T x = v^T A_x v \geqslant 0$ by positive semidefiniteness. Then $\alpha^T x = 0$ is a separating hyperplane for A.

4) Give an example for a linear program with no maximum (in other words, unbounded linear program) such that the corresponding integer program is not unbounded. **Solution:**

An example is to take the feasible region as a line through the origin that does not contain any integer point except for the origin (for example the line spanned by $(1, \sqrt{2})$). Then let the objective maximize the x-coordinate of a point on this line. As the line is unbounded, so is the LP but the integer program only contains the point $(0, 0)$ which has bounded objective value.

5) Let $E \subset \mathbb{R}^3$ be the ellipsoid $E = \{(x, y, z) : x^2 + \frac{y^2}{4} + \frac{z^2}{9} \leqslant 1\}$. Let H^+ be the half-space $H^+ = \{(x, y, z) : x + y + z \geqslant 0\}$. Find an ellipsoid E' such that $E' \supset E \cap H^+$ and $Vol(E') \leqslant Vol(E)e^{-1/(2(3+1))}$.

Solution:
The linear transformation $T(x, y, z) = (x, y/2, z/3)$ takes the ellipsoid E to the unit ball B.

Notice that T takes H^+ to the half-space $G^+ = \{x + 2y + 3z \geqslant 0\}$. Now we find a linear transformation R that will rotate the space about the origin such that B will remain fixed but G^+ will go to the half space $\{x \geqslant 0\}$. One way to do this is to take the vector $u = 1/\sqrt{14}(1, 2, 3)$ and complete it to an orthonormal basis: take also the vector $v = 1/\sqrt{5}(-2, 1, 0)$ and the vector w perpendicular to both u and v (you need to find it). Then consider the linear transformation R that takes u, v, w to $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$. This is the inverse of the matrix whose columns are u, v, w. The linear transformation $R(T())$ takes $E \cap H^+$ to the half ball $B \cap \{x \geqslant 0\}$. By the half-ball theorem, then the ellipsoid $F = \{\frac{16}{9}(x - \frac{1}{4})^2 + \frac{8}{9}y^2 + \frac{8}{9}z^2 \leqslant 1\}$ contains the half ball $B \cap \{x \geqslant 0\}$ and we also have $Vol(F) \leqslant e^{-1/(2(3+1))}Vol(B)$. If we now apply $T^{-1}(R^{-1}())$ on F we get the desired ellipsoid E'. One can get from here the algebraic description of E'. Technichally, we need to replace x, y, and z in the algebraic discription of F by $R(T(x, y, z))$.

6) (Bonus Question) Suppose we are given an oracle that tells us whether a polyhedron defined by $Ax \leqslant b$ is full-dimensional. Based on this oracle, this exercise develops a method to find an inequality $a^T x \leqslant \beta$ of $Ax \leqslant b$ that is satisfied by every feasible solution with *equality* in the case where $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ is not full-dimensional.

First we split $Ax \leqslant b$ into two systems $A_1 x \leqslant b_1$ and $A_2 x \leqslant b_2$, where $A_1 x \leqslant b_1$ are the inequalities that are satisfied with equality by every feasible $x^* \in \mathbb{R}^n$. The system $A_1 x \leqslant b_1$ is called the *implicit equalities* of $Ax \leqslant b$.

   i) If $Ax \leqslant b$ is feasible, then there exists a feasible solution x^* such that $A_2 x^* < b_2$ holds.

   ii) Argue that the implicit equalities of $A_1 x \leqslant b_1$ are $A_1 x \leqslant b_1$.

   iii) Suppose that $Ax \leqslant b$ is not full-dimensional and that $A'x \leqslant b'$ is full-dimensional, where $A'x \leqslant b'$ stems from $Ax \leqslant b$ by deleting one inequality $a^T x \leqslant \beta$. Show that $a^T x \leqslant \beta$ is an implicit equality of $Ax \leqslant b$.

   iv) If $a^T x \leqslant \beta$ is an implicit equality of $Ax \leqslant b$, then describe a feasibility problem in $\mathbb{R}^{n-1}$ that is equivalent to the one of $Ax \leqslant b$.