École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

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Discrete Optimization (Spring 2025)

Assignment 12

1) Show that the unit simplex $\Delta = conv\{0, e_1, \ldots, e_n\} \subset \mathbb{R}^n$ has volume $\frac{1}{n!}$.

2) Let $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ be a full dimensional 0/1 polytope and $c \in \mathbb{Z}^n$. We will show how we can use the ellipsoid method to solve the optimization problem $\{\max c^T x : x \in P\}$. Define $z^* := \max\{c^T x : x \in P\}$ and $c_{max} := \max\{|c_i| : 1 \leqslant i \leqslant n\}$.

   i) Show that P is contained inside the ball centered at $1/2 \cdot \vec{1}$ with radius $\sqrt{n}/2$ and give an upper bound on the volume of this ball.

   ii) Show that P contains a simplex of volume $1/n!$.

   iii) Scale down this simplex to obtain another simplex Δ' such that $P \cap (c^T x \geqslant \beta - 1/2)$ contains Δ' when nonempty. What is the volume of Δ'?

   iv) Show that the ellipsoid method requires $O(n^3 \log(n) c_{max})$ iterations to decide whether $P \cap (c^\top x \geqslant \beta - 1/2) = \emptyset$ for some integer β, i.e., use parts (i) and (iii) as suitable initial volume I_{init} and stopping volume L for the ellipsoid method. The ellipsoid method then terminates in $O(n \log(vol(I_{init})/L))$.

   v) Show that we can use binary search to find z^* with $\log(nc_{max})$ calls to the ellipsoid method.

   vi) Show how you can find an optimal solution x^* such that $c^T x^* = z^*$ in polynomial time.

3) Consider the complete graph G_n with 3 vertices, i.e., $G = (\{1, 2, 3\}, \binom{3}{2})$. Is the polyhedron of the linear programming relaxation of the vertex-cover integer program integral?