École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

May 13, 2025

Discrete Optimization (Spring 2025)

Assignment 12

1) Show that the unit simplex $\Delta = conv\{0, e_1, \ldots, e_n\} \subset \mathbb{R}^n$ has volume $\frac{1}{n!}$.

Solution:
We will solve the problem using induction on n. Starting with $n = 2$, we have that $\Delta = \Delta((0,0),(1,0),(0,1)) = conv\{0, e_1, e_2\}$, where $\Delta((0,0),(1,0),(0,1))$ means the triangle between the three vertices $(0,0),(1,0) and (0,1)$. It is clear that the volume of Δ is equal to 1/2. For the inductive step we use

$$\begin{aligned} vol_n(\Delta_n) &= \int_0^1 vol_{n-1}(x \cdot \Delta_{n-1})dx \\ &= \int_0^1 x^{n-1} vol_{n-1}(\Delta_{n-1})dx \\ &= \frac{1}{(n-1)!}\int_0^1 x^{n-1}dx \\ &= \frac{1}{n!} \end{aligned}$$

In this equation, we have use the result that, $vol_n(\lambda K) = \lambda^n vol_n(K)$, for any $K \subset \mathbb{R}^n$. We also have used the induction hypothesis in the third equation. Therefore, we get the desired result.

2) Let $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ be a full dimensional 0/1 polytope and $c \in \mathbb{Z}^n$. We will show how we can use the ellipsoid method to solve the optimization problem $\{\max c^T x : x \in P\}$. Define $z^* := \max\{c^T x : x \in P\}$ and $c_{max} := \max\{|c_i| : 1 \leqslant i \leqslant n\}$.

   i) Show that P is contained inside the ball centered at $1/2 \cdot \vec{1}$ with radius $\sqrt{n}/2$ and give an upper bound on the volume of this ball.
   ii) Show that P contains a simplex of volume $1/n!$.
   iii) Scale down this simplex to obtain another simplex Δ' such that $P \cap (c^T x \geqslant \beta - 1/2)$ contains Δ' when nonempty. What is the volume of Δ'?
   iv) Show that the ellipsoid method requires $O(n^3 \log(n) c_{max})$ iterations to decide whether $P \cap (c^\top x \geqslant \beta - 1/2) = \emptyset$ for some integer β, i.e., use parts (i) and (iii) as suitable initial volume I_{init} and stopping volume L for the ellipsoid method. The ellipsoid method then terminates in $O(n \log(vol(I_{init})/L))$.
   v) Show that we can use binary search to find z^* with $\log(nc_{max})$ calls to the ellipsoid method.
   vi) Show how you can find an optimal solution x^* such that $c^T x^* = z^*$ in polynomial time.

Solution:

   i) Note that all 0/1 polytopes are contained inside the ball centered at $1/2 \cdot \vec{1}$ with radius $\sqrt{n}/2$. We can upper bound the volume of this ball by $\sqrt{n}^n$.

ii) Since P is full dimensional it contains a simplex $\Delta = \text{conv}\{x_0, x_1, \ldots, x_n\}$ of volume $1/n!$.

iii) Let $P' := P \cap (c^T x \geqslant \beta - 1/2)$ and suppose that P' is nonempty. Take $x_0 \in P' \cap \{0,1\}^n$. We define $\alpha = \frac{1}{2nc_{max}}$ and consider the simplex $\Delta' = \text{conv}\{z_0, z_1, \ldots, z_n\}$ where $z_i = x_0 + \alpha(x_i - x_0)$. To see that Δ' is contained in P' we just need to check that each z_i is in P and satisfies $c^T z_i \geqslant \beta - 1/2$. Indeed, we have $c^T z_i = c^T x_0 + \alpha c^T(x_i - x_0) \geqslant \beta - 1/2$. Then we obtain that

$$vol(P') \geqslant vol(\Delta') \geqslant \frac{1}{n!}\left(\frac{1}{2nc_{max}}\right)^n.$$

iv) Observe that $P \cap (c^T x \geqslant \beta) = \emptyset$ iff $P' := P \cap (c^T x \geqslant \beta - 1/2) = \emptyset$, since the vertices of P are integral. So we want to lower bound the volume of P'. Let x_0 be an integral vertex in P'. The idea is to scale this simplex so that it is contained in P'. Setting $L = vol(\Delta')$ and using the fact that the ellipsoid method terminates in $O(n \log(vol(I_{init})/L))$ gives us the correct bound.

v) We use the fact that the value of $c^T x$ lies between $-nc_{max}$ and nc_{max} for any vertex x of P to conclude that z^* must also lie within these bounds. Using binary search on this interval of integer points takes $\log(nc_{max})$ steps.

vi) The algorithm described in (iv) and (v) gives us the optimal value z^* but also a point $y \in P$ such that $z^* \geqslant c^T y \geqslant z^* - 1/2$. We now take this point and project it onto the hyperplane $c^T x = z^*$. Let y' be the projection. If $y' \in P$ then we are done, otherwise we find a point on the line segment yy' that intersects a facet of P. We have now reduced the dimension of our problem by one and can proceed by once again projecting this new point onto the hyperplane $c^T x = z^*$. Continuing in this way we will arrive at an optimal solution in polynomial time.

3) Consider the complete graph G_n with 3 vertices, i.e., $G = (\{1,2,3\}, \binom{3}{2})$. Is the polyhedron of the linear programming relaxation of the vertex-cover integer program integral?

Solution:
The vertex-cover IP for the G_3 looks as follows:

$$\begin{aligned} \min \quad w_1x_1 + & w_2x_2 + w_3x_3 \\ & x1 + x2 \geqslant 1 \\ & x1 + x3 \geqslant 1 \\ & x2 + x3 \geqslant 1 \\ & x1, x2, x3 \in \mathbb{N}. \end{aligned}$$

Let $w_1 = w_2 = w_3 = 1$. Observe that we need at least two of the vertices in our vertex cover, i.e. the optimum of this integer program is 2. On the other hand, the vector $x = (1/2, 1/2, 1/2)$ is a feasible solution to the linear programming relaxation of objective value $3/2 < 2$. Assume that the polyhedron of the LP relaxation is integral. Thus, then the simplex algorithm will compute an optimal integer solution to the relaxation. As we have seen above, every integer solution has a value of at least 2, while an optimum fractional solution has a value of at less than $3/2$, a contradiction.