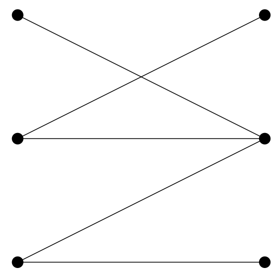
École Polytechnique Fédérale de Lausanne
Prof. Eisenbrand

May 6, 2025

Discrete Optimization (Spring 2025)

Assignment 11

1) Find a maximum cardinality matching and a minimum cardinality vertex cover in the following graph.



2) Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix and $b \in \mathbb{R}^n$ a vector. The ellipsoid $E(A,b)$ is defined as the image of the unit ball under the linear mapping $t(x) = Ax + b$. Show that

$$E(A,b) = \{x \in \mathbb{R}^n : (x-b)^\top A^{-\top} A^{-1}(x-b) \leqslant 1\}.$$

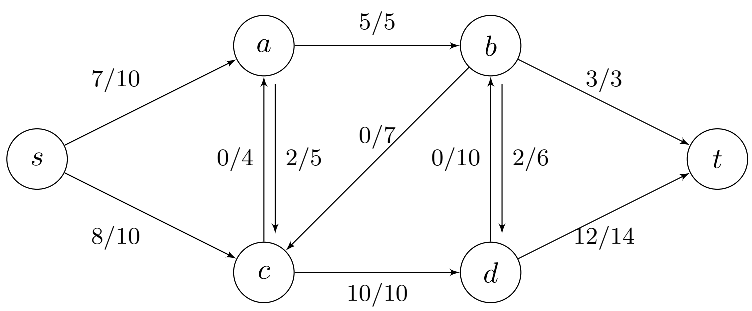
3) Draw $E(A,b)$ for $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. What are the axes of $E(A,b)$?

4) Let $D = (V,A)$ be a directed graph and $A_D \in \{0, \pm 1\}^{|V|\times|A|}$ be the node-edge incidence matrix of D. Assume that the underlying undirected graph $G = (V,E)$ with $E = \{uv\colon uv \in A \text{ or } vu \in A\}$ is connected.

   i) Show that any row of A_D is in the span of the other rows.

   ii) Let $T \subseteq A$ be a selection of $n-1$ arcs of A such that the induced undirected graph is a spanning tree of G. Show that the corresponding columns of A_D are linearly independent.

5) Let $f \in \mathbb{R}^{|A|}_{\geqslant 0}$ be a flow of a directed graph. Show that we can find a feasible flow f^* such that $f^* = \sum_{p\in P} \mu_p \cdot p + \sum_{c\in C} \mu_c \cdot c$ where C is a set of cycles in the graph, P is a set of paths in the graph, and $\mu_l, \mu_p \in \mathbb{R}_{\geqslant 0}$.



Example: This flow can be decomposed into the following combination of paths:
•$p1 : s \to a \to b \to t$ ($f1$ assigns 3 units to each edge in $p1$)

•$p2 : s \to c \to d \to t$ ($f2$ assigns 8 units to each edge in $p2$)
•$p3 : s \to a \to b \to d \to t$ ($f3$ assigns 2 units to each edge in $p3$)
•$p4 : s \to a \to c \to d \to t$ ($f4$ assigns 2 units to each edge in $p4$)

6) Let $D = (V, A)$ be a digraph. For every $a \in A$, let $l_a, u_a \in \mathbb{R}_{\geqslant 0}$ be given such that $l_a \leqslant u_a$. Show that the set of circulations $\{x \in \mathbb{R}^A : A_D x = 0, l \leqslant x \leqslant u\}$ (with A_D being the node-arc incidence matrix of D) is nonempty if and only if

$$\sum_{a \in \delta^-(X)} l_a \leqslant \sum_{a \in \delta^+(X)} u_a \quad \text{for all } X \subseteq V.$$