École Polytechnique Fédérale de Lausanne
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April 29, 2025

Discrete Optimization (Spring 2025)

Assignment 10

1) Let $M \in \mathbb{Z}^{n\times m}$ be totally unimodular. Prove that the following matrices are totally unimodular as well.

   (a) M^T

   (b) $\begin{pmatrix} M & I_n \end{pmatrix}$

   (c) $\begin{pmatrix} M & -M \end{pmatrix}$

   (d) $M \cdot \left(I_n - 2e_j e_j^T\right)$ for any $j \in [n]$.

2) A family $\mathcal{F}$ of subsets of a finite groundset E is laminar, if for all $C, D \in \mathcal{F}$, one of the following holds:

   (a) $C \cap D = \emptyset$

   (b) $C \subseteq D$

   (c) $D \subseteq C$.

   Let F_1 and F_2 be two laminar families of the same groundset E and consider its union $F_1 \cup F_2$. Define the $|F_1 \cup F_2| \times |E|$ adjacency matrix A as follows: For $F \in F_1 \cup F_2$ and $e \in E$ we have $A_{F,e} = 1$, if $e \in F$ and $A_{F,e} = 0$ otherwise.
   Show that A is totally unimodular.

3) Let G be a graph and let A be its node-edge incidence matrix. We have seen that if G is bipartite then A is totally unimodular. Prove the converse, i.e., if A is totally unimodular then G is bipartite.

4) Given a graph $G = (V, E)$, the spanning tree polytope $PST(G)$ is defined as follows:

$$PST(G) = \{x \in \mathbb{R}^E : x(E(U)) \leqslant |U| - 1 \ \forall U \subset V, x(E) = |V| - 1, x \geqslant 0\}.$$

   We will show that each vertex of $PST(G)$ is integral (i.e. $PST(G)$ is the convex hull of the incidence vectors of the spanning trees of G) by an uncrossing argument. Given x^* a vertex of $PST(G)$, let $F = \{U \subset V : x^*(E(U)) = |U| - 1\}$.

   (a) Let $A, B \in F$, show that $A \cap B, A \cup B \in F$.

   (b) Show that if L is a maximal laminar subfamily of F, then $span(L) = span(F)$ (where $span(F) = span\{\chi^{E(A)}, A \in F\}$, and similarly for L).