École Polytechnique Fédérale de Lausanne
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Discrete Optimization (Spring 2025)

Assignment 9

1) Let $M \in \mathbb{Z}^{n\times m}$ be totally unimodular. Prove that the following matrices are totally unimodular as well.

   (a) M^T

   (b) $\begin{pmatrix} M & I_n \end{pmatrix}$

   (c) $\begin{pmatrix} M & -M \end{pmatrix}$

   (d) $M \cdot \left(I_n - 2e_j e_j^T\right)$ for any $j \in [n]$.

   Solution:

   (a) Let A be a square submatrix of M^T . Then $\det(A) = \det(A^T) \in \{-1, 0, 1\}$ as A^T is a square submatrix of M and M is totally unimodular.

   (b) Let A be a square submatrix of $(M\ I_n)$. Let $a_1, ..., a_k$ be the columns of A that originate from I_n . Hence, each of these columns has as most one 1-entry, the other entries are 0. Hence, using Laplace-expansion successively along these columns we get that $|\det(A)| = |\det(A')|$ for some square submatrix of M . Since M is TU, this shows $\det(A) \in \{-1, 0, 1\}$.

   (c) Let A be a square submatrix of $\begin{pmatrix} M & -M \end{pmatrix}$. Let $a_1, ..., a_k$ be the columns of A that originate from $-M$. Let A' be the matrix obtained from A by multiplying $a_1, ..., a_k$ with -1. Hence $|\det(A)| = |det(A')|$. Now we distinguish two cases. Case 1: A' is (up to permutation of columns) a square submatrix of M . Since M is TU, we have $\det(A') \in \{-1, 0, 1\}$. Case 2: A' has at least two identical columns. Hence $\det(A') = 0$. We conclude that in both cases we have $\det(A) \in \{-1, 0, 1\}$.

   (d) Observe that $M \cdot (I_n - 2e_j e_j^T)$ is obtained from M by multiplying one column with -1. Thus, $M \cdot (I_n - 2e_j e_j^T)$ is (up to permutation of columns) a submatrix of $\begin{pmatrix} M & -M \end{pmatrix}$. Thus this matrix is also TU.

2) A family $\mathcal{F}$F of subsets of a finite groundset E is laminar, if for all $C, D \in \mathcal{F}$, one of the following holds:

   (a) $C \cap D = \emptyset$

   (b) $C \subseteq D$

   (c) $D \subseteq C$.

   Let F_1 and F_2 be two laminar families of the same groundset E and consider its union $F_1 \cup F_2$. Define the $|F_1 \cup F_2| \times |E|$ adjacency matrix A as follows: For $F \in F_1 \cup F_2$ and $e \in E$ we have $A_{F,e} = 1$, if $e \in F$ and $A_{F,e} = 0$ otherwise.
   Show that A is totally unimodular.

   Solution:
   Let F_1 and F_2 be two laminar families on the same groundset E , and A the corresponding adjacency matrix. Observe that every square submatrix of A also is the adjacency matrix of

two laminar families: Removing a row from A corresponds to deleting a set from the laminar families. Removing a column from A corresponds to removing an element of the ground- set from all sets of the laminar families. Both operations preserve the structure of laminar families.

For that reason, it is sufficient to show the following statement: Every square matrix A that is adjacency matrix of two laminar families has determinant ± 1 or 0. We will transform A with elementary column operations as follows: Let $e \in E$ be an element from the groundset that is contained in at least two sets from F_1. Let $S_1, ..., S_k$ be the sets of F_1 with $e \in S_i$. Using the properties of laminar families, we know that there is a $l \in \{1, ..., k\}$ such that $S_l \subseteq S_i$ for all $i = 1, ..., k$. Redefine $S_i := S_i \setminus S_l$ for all $i \neq l$. Observe that F_1 is still a laminar family after this modification. Also observe that the operation of removing the set S_l corresponds to subtraction the row S_l from the other rows S_i in the matrix A. Hence we can apply this transformation until each $e \in E$ is contained in at most one set of F_1. Similarly we can apply this transformation to F_2 until each $e \in E$ is contained in at most one set of F_2. Applying the corresponding elementary row operations to A yields a matrix A' with $\det(A) = \det(A')$. A' has the property that there are two disjoint subsets of rows, the rows corresponding to F_1 and the rows corresponding to F_2, such that each column of A' has at most one 1-entry in the rows of F_1 and at most one 1-entry in the rows of F_2. All other entries are 0.
Let A'' be the submatrix of A' consisting only of the columns with two 1-entries. Note that this is a node-edge incidence matrix of a bipartite graph. Hence A'' is TU. With Exercise 1.2 we get that A' is TU. Hence $\det(A) \in \{-1, 0, 1\}$.

3) Let G be a graph and let A be its node-edge incidence matrix. We have seen that if G is bipartite then A is totally unimodular. Prove the converse, i.e., if A is totally unimodular then G is bipartite.

Solution:
Let the incidence matrix of G be totally unimodular and assume towards contradiction that G is not bipartite. Then G must contain a cycle of odd length. Let this cycle contain some vertices of G $\{v_1, \ldots, v_{2k+1}\}$ for some $k \in \mathbb{N}$. Let the edges of this cycle be $\{e_1, \ldots, e_{2k+1}\}$. Now, consider the submatrix of A indexed by $[v_1, \ldots, v_{2k+1}] \times [e_1, \ldots, e_{2k+1}]$. Then this submatrix of the cycle (up to permutation of the columns) looks as follows:

	e_1	e_2	e_3	$\ldots$	e_{2k+1}	
v_1	1	0	0	$\ldots$	0	1
v_2	1	1	0	$\ldots$	0	0
v_3	0	1	1	$\ldots$	0	0
$\vdots$		$\ddots$				
v_{2k+1}	0	0	0	$\ldots$	1	1

Then since the number of rows and columns is odd, we can do row reduction and end with one row that has a value 2 giving the whole submatrix a determinant of 2. This means A is not submodular, a contradiction.

4) Given a graph $G = (V, E)$, the spanning tree polytope $PST(G)$ is defined as follows:

$$PST(G) = \{x \in \mathbb{R}^E : x(E(U)) \leqslant |U| - 1 \; \forall U \subset V, x(E) = |V| - 1, x \geqslant 0\}.$$

We will show that each vertex of $PST(G)$ is integral (i.e. $PST(G)$ is the convex hull of the incidence vectors of the spanning trees of G) by an uncrossing argument. Given x^* a vertex of $PST(G)$, let $F = \{U \subset V : x^*(E(U)) = |U| - 1\}$.

(a) Let $A, B \in F$, show that $A \cap B, A \cup B \in F$.

(b) Show that if L is a maximal laminar subfamily of F, then $span(L) = span(F)$ (where $span(F) = span\{\chi^{E(A)}, A \in F\}$, and similarly for L).

Solution:

(a) We have:

$$|A| - 1 + |B| - 1 = x * (E(A)) + x * (E(B)) \leqslant x * (E(A \cup B)) + x * (E(A \cap B))$$

where the inequality follows since the edges in $E(A \cap B)$ are counted twice and each other edge induced by A or B is also induced by $A \cup B$. Now,

$$x * (E(A \cup B)) + x * (E(A \cap B)) \leqslant |A \cup B| - 1 + |A \cap B| - 1 = |A| - 1 + |B| - 1$$

hence all the inequalities hold with equality and in particular $x * (E(A \cup B)) = |A \cup B| - 1$ and $x * (E(A \cap B)) = |A \cap B| - 1$.

(b) Similarly as in the proof seen before, for $A \in F$ we define $viol(A) = \{B \in L : A, B \text{ are intersecting}\}$. Assume by contradiction that $span(L)$ is a strict subset of $span(F)$, and let A such that $\chi^A \in span(F) \setminus span(L)$ and $|viol(A)|$ is minimum. By maximality of L, $|viol(A)| \geqslant 1$ otherwise $L \cup A$ would be a larger laminar family contained in F. Hence let $B \in viol(A)$, we claim that $|viol(A \cap B)| < |viol(A)|$. Indeed, let $C \in viol(A \cap B), C \neq B$, we have that $C \setminus A \cap B, A \cap B \setminus C, A \cap B \cap C$ are non-empty. Moreover, $C \in L$, hence either $C \subset B$, or $B \subset C$ or $B \cap C = \emptyset$. The last one is not possible as $A \cap B \cap C \subset B \cap C$. So assume $C \subset B$: then if $C \subset A$, $C \subset A \cap B$, a contradiction to $C \setminus A \cap B$ being non-empty. If $A \subset C$, then $A \subset B$, a contradiction to $B \in viol(A)$. If $A \cap C = \emptyset$, then we get again a contradiction to $A \cap B \cap C$ being non-empty. Hence in this case A, C are intersecting and the claim is proved. In the case $B \subset C$, the claim is proved similarly. With analogous arguments one proves that $|viol(A \cup B)| < |viol(A)|$. Now, by minimality of $|viol(A)|$, we must have that $\chi^{E(A \cup B)}, \chi^{E(A \cap B)} \in span(L)$, but then $\chi^{E(A)} = \chi^{E(A \cup B)} + \chi^{E(A \cap B)} - \chi^{E(B)} \in span(L)$, a contradiction. (Notice that the equality holds because $A, B \in F$ as seen in the proof of part 1).

We remark that we are now able to conclude that the vertices of $PST(G)$ are integral. In particular, $x*$ is the unique solution of the system $x(E(U)) = |U| - 1$ for

$$x(E(U)) = |U| - 1 \quad \forall U \in F$$
$$x_e = 0 \quad \forall e \in \bar{E}$$

for some $\bar{E} \subset E$. Using this argument, we can reduce the system to

$$x(E(U)) = |U| - 1 \quad \forall U \in L$$
$$x_e = 0 \quad \forall e \in \bar{E}$$

Now, the matrix associated to the system is totally unimodular, hence x^* is an integer vector.