Discrete Optimization – Grading Scheme

Award points only if arguments are complete. No half points are to be given.

Open Questions Grading Scheme

Question 15 (8 points)

a) Let $P = \{x \in \mathbb{R}^n \colon Ax \leq b\} \subseteq \mathbb{R}^n$ be a bounded non-empty polyhedron. Then $\text{rank}(A) = n$ and P has vertices (no line).

(+1 point)

Let $v_1, \dots, v_k \in \mathbb{R}^n$ be the list of vertices of P. Claim: $P = \text{conv}(v_1, \dots, v_k)$.

$P \supseteq \text{conv}(v_1, \dots, v_k)$ since P is convex. (+1 point)

Suppose $P \supset \text{conv}(v_1, \dots, v_m)$ and let $x^* \in P \setminus \text{conv}(v_1, \dots, v_m)$. Separation theorem implies there exists $c \in \mathbb{R}^n, \beta \in \mathbb{R}$ with

$$c^T x^* > \beta \text{ and } c^T v_i < \beta, \quad i = 1, \dots, m.$$

(+1 point)

Optimum of $\max\{c^T x \colon x \in P\} > \beta$ and is attained at a vertex. Contradiction! (+1 point)

b) The LP

$$\begin{aligned} \min \quad & 0^T \lambda \\ \lambda_1 v_1 + \cdots + \lambda_m v_m &= x^* \\ \lambda_1 + \cdots + \lambda_m &= 1 \\ \lambda &\geq 0 \end{aligned}$$

is infeasible.

Its dual is the LP:

$$\begin{aligned} \max \left(\begin{smallmatrix} x^* \\ 1 \end{smallmatrix} \right)^T y & \\ \left(\begin{smallmatrix} v_i \\ 1 \end{smallmatrix} \right)^T y \leq 0 & \quad i = 1, \dots, m \end{aligned}$$

(+1 point)

The dual is feasible ($y^* = 0$) and unbounded $\max = +\infty$ (Duality Theorem)

(+1 point)

By adding the inequalities $-1 \leq y \leq 1$ the dual remains feasible. (Scale a feasible $y^* \neq 0$ with $1/\|y^*\|_\infty$.) The optimal value is > 0 and attained at a basic feasible solution y_B^* and there are only finitely many.

(+1 point)

Conclusion: There exists a finite set of vectors $y_1, \dots, y_k \in \mathbb{R}^{n+1}$ with

$$\sum_{i=1}^{n} (y_j)_i x_i \leq -(y_j)_{n+1}$$

valid for $v_1, \dots, v_n$ for each j. And for each $x^* \notin \operatorname{conv}(x_1, \dots, x_m)$ there exists an i such that

$$\sum_{i=1}^{n} (y_j)_i x_i^* > -(y_j)_{n+1}$$

This set of inequalities defines therefore $\operatorname{conv}(v_1, \dots, v_m)$. (+1 point)

Question 16 (6 points)

a) For $\|x\| \leq 1$ one has $\sum_{i=2}^{n} x_i^2 \leq 1 - x_1^2$

(+1 point)

and thus

$$\begin{aligned} \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2 \\ \leq \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}(1 - x_1^2) \end{aligned} \quad (1)$$

This shows that $x \in E$ if x is contained in the half-ball and $x_1 = 0$ or $x_1 = 1$.

(+1 point)

Consider right-hand-side of (1) as a function of x_1, i.e., consider

$$f(x_1) = \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}(1 - x_1^2).$$

The first derivative is

$$f'(x_1) = 2 \cdot \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right) - 2 \cdot \frac{n^2-1}{n^2} x_1.$$

(+1 point)

We have $f'(0) < 0$ and since both $f(0) = 1$ and $f(1) = 1$ (and since $f(x_1)$ is a 2-degree polynomial w.r.t. x_1), we have $f(x_1) \leq 1$ for all $0 \leq x_1 \leq 1$ and the assertion follows. (+1 point)

b) E is described as $E = \{x \in \mathbb{R}^n \mid \|A^{-1}x - A^{-1}b\|\}$, where A is the diagonal matrix with diagonal entries

$$\frac{n}{n+1}, \sqrt{\frac{n^2}{n^2-1}}, \dots, \sqrt{\frac{n^2}{n^2-1}}$$

and b is the vector $b = (1/(n+1), 0, \dots, 0)$.

E is thus the image of V_n under $t(x) = Ax + b$.

The determinant of A is $\frac{n}{n+1}\left(\frac{n^2}{n^2-1}\right)^{(n-1)/2}$ and therefore

$$\mathrm{Vol}(E) = \frac{n}{n+1}\left(\frac{n^2}{n^2-1}\right)^{(n-1)/2} \cdot V_n.$$

(+1 point)

Since $1 + x \leq e^x$ this is bounded by

$$e^{-1/(n+1)} e^{(n-1)/(2\cdot(n^2-1))} = e^{-\frac{1}{2(n+1)}}.$$

(+1 point)

Question 17 (4 points)

$\{A_1 x \leq b_1\} \subseteq \{A_2 x \leq b_2\}$ if and only if:

For each ineq. $a^T x \leq \beta$ of $A_2 x \leq b_2$:

$$\max\{a^T x \colon x \in \mathbb{R}^n,\ A_1 x \leq b_1\} \leq \beta$$

(+2 point)

Algorithm to decide $\{A_1 x \leq b_1\} \subseteq \{A_2 x \leq b_2\}$:

For each inequality $a^T x \leq \beta$ of $A_2 x \leq b_2$ solve the linear program

$$\max\{a^T x \colon x \in \mathbb{R}^n,\ A_1 x \leq b_1\}.$$

(+1 point)

If $\max \leq \beta$ holds every time, then assert $\{A_1 x \leq b_1\} \subseteq \{A_2 x \leq b_2\}$

Otherwise, assert $\{A_1 x \leq b_1\} \nsubseteq \{A_2 x \leq b_2\}$

(+1 point)

Question 18 (8 points)

a) Take an optimal flow described by the linear program, and decompose it into cycles using the flow decomposition theorem. (+1 point)
Suppose there is a flow cycle, that is not the minimum mean cycle, and has length k_1 and the sum of c_{ij}'s is c_1. Let f be the sum of values of flow over this cycle (f/k_1 per edge). This implies that this cycle contributes $c_1 \cdot f/k_1$ to the value of the objective. Let the minimum mean cycle have some length k_2, and the sum of c_{ij}'s over this cycle is c_2. Obviously, $c_2/k_2 < c_1/k_1$. Then if we took flow value f , and placed it throughout the minimum mean cycle, this flow would contribute $c_2 \cdot f/k_2$ to the objective, and would be a smaller objective value. Therefore, in the optimum solution all flow goes through the minimum mean cycles. (+2 points)

b) The dual is following:

$$\begin{aligned} &\max \lambda \\ &\lambda + p_i - p_j \leq c_{ij}\ \forall i, j. \end{aligned}$$

(+1 point)