Teacher: Prof. Friedrich Eisenbrand
Discrete Optimization -
21 June 2025
Duration: 180 minutes

David Hilbert

SCIPER : **2** Signature :

Do not turn the page before the start of the exam. This document is double-sided, has 24 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is not permitted during the exam.
- First part : for the **multiple choice** questions, we give :

  +2 points if your answer is correct,
  0 points if you give no answer or more than one,
  −1 points if your answer is incorrect.

- Second Part : for the **true/false** questions, we give :

  +1 points if your answer is correct,
  0 points if you give no answer,
  −1 points if your answer is incorrect.

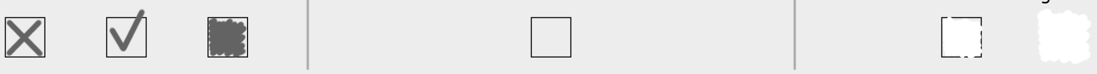
- Third part : for the **open questions**, the number of points is noted above each question. Leave the checkbox empty. **Use the grid** provided for your response. Each of these grids has a **reserve version** in the corresponding section. We will only consider one grid. If two have been used, **cross out** the one that is not to be evaluated. Any incorrect statement in your text will result in a one-point **deduction**.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien

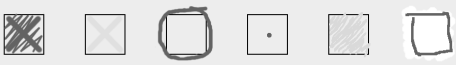
choisir une réponse | select an answer
Antwort auswählen

ne PAS choisir une réponse | NOT select an answer
NICHT Antwort auswählen

Corriger une réponse | Correct an answer
Antwort korrigieren



ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte



First part: multiple choice questions

For each question, mark the box corresponding to the correct answer.

Question [`MCQ-01`] Consider the following linear program

$$\begin{aligned} \max \quad & 3x_1 + x_2 \\ s.t. \quad & 2x_1 + x_2 + x_3 = 2 \\ & -x_1 + 2x_2 + x_4 = 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_3 \geq 0 \\ & x_4 \geq 0. \end{aligned}$$

Which of the following is equivalent to the dual of this linear program?

■

$$\begin{aligned} \min \ & 2y_1 + y_2 \\ s.t. \ & 2y_1 - y_2 \geq 3 \\ & y_1 + 2y_2 \geq 1 \\ & y_1 \geq 0 \\ & y_2 \geq 0. \end{aligned}$$

□

$$\begin{aligned} \min \ & 2y_1 + y_2 \\ s.t. \ & 2y_1 - y_2 \leq 3 \\ & y_1 + 2y_2 \leq 1 \\ & y_1 \geq 0 \\ & y_2 \geq 0. \end{aligned}$$

□

$$\begin{aligned} \min \ & 2y_1 + y_2 \\ s.t. \ & 2y_1 - y_2 \geq 3 \\ & y_1 + 2y_2 \geq 1. \end{aligned}$$

□ None of the others is correct.

Question [`MCQ-02`] Consider the following matrix game defined by

$$\begin{pmatrix} 5 & -1 \\ 2 & 4 \end{pmatrix}$$

Which is an optimal mixed strategy $y = (y_1, y_2) \in \mathbb{R}^2$ for the column player?

□ $(0, 1)$
■ $(5/8, 3/8)$
□ $(1, 0)$
□ $(3/10, 7/10)$
□ None of these other choices.

Question [MCQ-03] Consider the following linear program

$$\begin{aligned} \max \quad & x - 4y + 2z \\ \text{s.t.} \quad & 2x - y - 2z \leq 4 \\ & -2x + 3y + 3z \leq 5 \\ & -x + 2y + z \leq 1 \\ & x \geq 0 \\ & y \geq 0 \\ & z \geq 0 \end{aligned}$$

Solve this linear program using the Simplex algorithm with the *smallest index rule* and initial basis $\{4, 5, 6\}$, which corresponds to the vertex $(0, 0, 0) \in \mathbb{R}^3$. Which of the following is the path of bases returned by the Simplex algorithm?
Here are the inverse matrices of all the feasible bases.

$$B = \{1,2,3\} \Rightarrow A_B^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1/3 & 0 & 2/3 \\ 1/3 & 1 & -4/3 \end{pmatrix}, \quad B = \{1,2,5\} \Rightarrow A_B^{-1} = \begin{pmatrix} 3/2 & 1 & 3/2 \\ 0 & 0 & -1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$B = \{1,3,6\} \Rightarrow A_B^{-1} = \begin{pmatrix} 2/3 & 1/3 & -1 \\ 1/3 & 2/3 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \{1,5,6\} \Rightarrow A_B^{-1} = \begin{pmatrix} 1/2 & -1/2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$B = \{2,3,5\} \Rightarrow A_B^{-1} = \begin{pmatrix} 1 & -3 & -3 \\ 0 & 0 & -1 \\ 1 & -2 & -1 \end{pmatrix}, \quad B = \{3,4,5\} \Rightarrow A_B^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$B = \{3,4,6\} \Rightarrow A_B^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1/2 & -1/2 & 1/2 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \{4,5,6\} \Rightarrow A_B^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- ■ $\{4,5,6\} \to \{1,5,6\} \to \{1,2,5\}$
- □ $\{4,5,6\} \to \{3,4,5\} \to \{2,3,5\} \to \{1,2,3\}$
- □ $\{4,5,6\} \to \{1,5,6\} \to \{1,3,6\}$
- □ $\{4,5,6\} \to \{3,4,6\} \to \{3,4,5\}$
- □ None of the others is the path.

Question [MCQ-04] Which of the following statements about totally unimodular matrices is *false*?

- □ For any complete bipartite graph, G, the incidence matrix A of G is TU.
- ■ Let A be a TU matrix and let B be a $k \times k$ submatrix of A where all columns of B have exactly two nonzero entries. Then it must be that $\mathbf{1}^T B = 0$ and $\det(B) = 0$.
- □ Let A be an $n \times n$ TU matrix. Let B be the matrix $A_{[1,k],[1,k]}$ which is the submatrix of A containing the first k rows and k columns. Then B^{-1} is an integer matrix.
- □ Let A be a TU matrix. Then the matrix $[A \;\; -A \;\; I \;\; -I]$ is also a TU matrix where I is the identity matrix.

Question [MCQ-05] Consider the linear program (I)

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b, \\ & 0 \leq x_i \leq D, \quad \text{for all } i = 1, \dots, n, \\ & x \in \mathbb{R}^n, \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n, D \in \mathbb{R}_{>0}$ and $\operatorname{rank}(A) = n$. Suppose that the optimal solution of this linear program (I) is attained at $y \in \mathbb{R}^n$. Choose an index i from $\{1, \dots, m\}$ uniformly at random, i.e., with probability $1/m$. Remove the corresponding inequality $a_i^T x \leq b_i$ from $Ax \leq b$ to get $A'x \leq b'$. Consider the linear program (II)

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & A'x \leq b', \\ & 0 \leq x_i \leq D, \quad \text{for all } i = 1, \dots, n, \\ & x \in \mathbb{R}^n. \end{aligned}$$

Suppose that the optimal solution of this linear program (II) is attained at $y' \in \mathbb{R}^n$. Which of the following must be correct?

- [ ] None of the other statements is correct.
- [ ] $\Pr[c^T y < c^T y'] \geq 1/m$.
- [ ] $\Pr[c^T y = c^T y'] = D/m$.
- [x] $\Pr[c^T y < c^T y'] \leq n/m$.

Question [MCQ-06] Consider the problem of finding a feasible solution of $Ax = b$ with the minimum infinity norm: $\min\{\|x\|_\infty : Ax = b, x \in \mathbb{R}^n\}$ where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Which of the following is a correct linear programming formulation of this problem?

■

$$\begin{aligned} \min \quad & \beta \\ \text{s.t.} \quad & Ax = b, \\ & z_i \geq x_i, \quad \text{for all } i = 1, \ldots, n \\ & z_i \geq -x_i, \quad \text{for all } i = 1, \ldots, n \\ & z_i \leq \beta, \quad \text{for all } i = 1, \ldots, n \\ & x \in \mathbb{R}^n, z \in \mathbb{R}^n, \beta \in \mathbb{R} \end{aligned}$$

□

$$\begin{aligned} \min \quad & \beta \\ \text{s.t.} \quad & Ax = b, \\ & z_i \geq x_i, \quad \text{for all } i = 1, \ldots, n \\ & z_i \geq -x_i, \quad \text{for all } i = 1, \ldots, n \\ & \sum_i^n z_i \leq \beta, \\ & x \in \mathbb{R}^n, z \in \mathbb{R}^n, \beta \in \mathbb{R} \end{aligned}$$

□

$$\begin{aligned} \min \quad & \beta \\ \text{s.t.} \quad & Ax = b, \\ & x_i \leq \beta, \quad \text{for all } i = 1, \ldots, n \\ & -x_i \leq \beta, \quad \text{for all } i = 1, \ldots, n \\ & x \in \mathbb{R}^n, \beta \in \mathbb{R} \end{aligned}$$

□

$$\begin{aligned} \min \quad & \beta \\ \text{s.t.} \quad & Ax = b, \\ & \sum_{i=1}^n x_i \leq \beta \\ & x \in \mathbb{R}^n, \beta \in \mathbb{R} \end{aligned}$$

□

$$\begin{aligned} \min \quad & \beta \\ \text{s.t.} \quad & Ax = b, \\ & x_i \leq \beta, \quad \text{for all } i = 1, \ldots, n \\ & x \in \mathbb{R}^n, \beta \in \mathbb{R} \end{aligned}$$

Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question [TF-01] The union of two polyhedra is also a polyhedron.

□ TRUE ■ FALSE

Question [TF-02] Let $x^* \in \mathbb{R}^n$ be an optimal solution of the linear program $\max\{c^T x : Ax \leq b\}$ where $A \in \mathbb{R}^{m\times n}, c \in \mathbb{R}^n$. A constraint $a_i^T x \leq b_i$ of $Ax \leq b$ is *tight* at x^* if $a_i^T x^* = b_i$ holds. Let I be the set of indices of all the tight inequalities of $Ax \leq b$ at x^*.
Then $c \in \text{cone}(\{a_i\}_{i\in I})$.

■ TRUE □ FALSE

Question [TF-03] For any finite subset $X \subseteq \mathbb{R}^n$, there exists $\tilde{X} \subseteq X$ which is linearly independent and of size $\leq n$, such that $\text{cone}(\tilde{X}) = \text{cone}(X)$.

□ TRUE ■ FALSE

Question [TF-04] Consider the simplex algorithm with an unspecified pivoting rule, i.e., the first version of the simplex algorithm we saw in class. An index that has just entered the basis B can leave B in the very next iteration.

□ TRUE ■ FALSE

Question [TF-05] Consider two polyhedra $P = \{x : Ax \leq b\}$ and $P' = \{x : A'x \leq b'\}$ where $A, A' \in \mathbb{R}^{m\times n}$ and $b, b' \in \mathbb{R}^m$. If $P \cap P' = \emptyset$, then there exists $y, z \in \mathbb{R}^m_{\geq 0}$ such that $y^T A + z^T A' = 0$ but $y^T b + z^T b' < 0$.

■ TRUE □ FALSE

Question [TF-06] Consider the simplex algorithm with an unspecified pivoting rule, i.e., the first version of simplex algorithm we saw in class. An index that has just left the basis B can enter in the very next iteration.

■ TRUE □ FALSE

Question [TF-07] Consider the linear program $\max\{c^T x : Ax \leq b\}$ with $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ and let y^* be a feasible solution of its dual. Then for any given $\alpha \in \mathbb{R}^n, \beta \in \mathbb{R}$, there exists $y_0 \in \mathbb{R}$ such that (y^*, y_0) is also a feasible solution of the dual of $\max\{c^T x : Ax \leq b, \alpha^T x \leq \beta\}$.

■ TRUE □ FALSE

Question [TF-08] There exists a linear program $\max\{c^T x : Ax \leq b\}$, with $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ such that both this linear program and its dual are unbounded.

□ TRUE ■ FALSE

Third part, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 15: *This question is worth 8 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6 ☐ 7 ■ 8

In this question, we will prove that a set $P \subseteq \mathbb{R}^n$ is the convex hull $P = \text{conv}(v_1, \dots, v_m)$ of some points $v_1, \dots, v_m \in \mathbb{R}^n$ if and only if $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ is a bounded polyhedron.

(a) Prove that if P is a bounded polyhedron, then $P = \text{conv}(v_1, \dots, v_m)$ for some vectors $v_1, \dots, v_m \in \mathbb{R}^n$.

Hint: Consider the vertices of $\{Ax \leq b\}$ (are there any?). If convex hull of these is not the entire polytope, separation theorem and a certain optimization problem leads to a contradiction.

(b) Prove that if $P = \text{conv}(v_1, \ldots, v_m)$ for some vectors $v_1, \ldots, v_m \in \mathbb{R}^n$, then P is a bounded polyhedron.

Hint: For an $x^* \notin P$, the linear program

$$\begin{aligned} \min \quad 0^T \lambda \\ \lambda_1 v_1 + \cdots + \lambda_m v_m &= x^* \\ \lambda_1 + \cdots + \lambda_m &= 1 \\ \lambda &\geq 0 \end{aligned}$$

is infeasible. Consider the dual and show that it is feasible. Make it bounded and use the fact that the resulting LP has only a finite number of basic feasible solutions.

Reserve space. Check the description on the first page for its use.

(a)

(b)

Question 16: *This question is worth 6 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ■ 6

Prove the following two statements.

(a) The half-ball $H = \{x \in \mathbb{R}^n \mid \|x\| \leq 1, x_1 \geq 0\}$ is contained in the ellipsoid

$$E = \left\{ x \in \mathbb{R}^n \mid \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2 \leq 1 \right\}$$

(b) The volume of the ellipsoid E defined in (a) is bounded by $e^{-\frac{1}{2(n+1)}} \cdot V_n$, where V_n is the volume of the unit ball in $\mathbb{R}^n$. *Hint: For any* $x \in \mathbb{R}$, $1 + x \leq e^x$.

Reserve space. Check the description on the first page for its use.

(a)

(b)

Question 17: *This question is worth 4 points.*

☐ 0 ☐ 1 ☐ 2 ☐ 3 ■ 4

Let $A_1 \in \mathbb{R}^{m_1 \times n}, A_2 \in \mathbb{R}^{m_2 \times n}$ and $b_1 \in \mathbb{R}^{m_1}, b_2 \in \mathbb{R}^{m_2}$. Describe (in words, no pseudo-code) an algorithm, based on linear programming that decides whether $\{x \in \mathbb{R}^n \colon A_1 x \leq b_1\} \subseteq \{x \in \mathbb{R}^n \colon A_2 x \leq b_2\}$. Justify correctness of your algorithm.

Catalog

Reserve space. Check the description on the first page for its use.

Question 18: *This question is worth 4 points.*

□ 0 □ 1 □ 2 □ 3 ■ 4

Consider the complete directed graph $G = (V, A)$ on n vertices with a cost function $c : A \to \mathbb{R}$ where each arc $(i, j) \in A$ has cost c_{ij}. Note that the costs are possibly negative. Suppose that you want to find a minimum *mean cycle* in G, which is a cycle with the minimum ratio of cost to length (number of edges) of the cycle. Going around such a cycle repeatedly (assuming it is negative) provides you with the maximum possible profit per unit length/time, so is the fastest way to earn money if you are, for example, a delivery service. Minimum mean cycle also arises as a subroutine for solving problems like min cost flow.

Consider the following linear program:

$$\begin{aligned} \min \; & \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} f_{ij} \\ s.t. \; & \sum_{j} f_{ij} - f_{ji} = 0, \; \forall\, i = 1, \ldots, n, \\ & \sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij} = 1, \\ & f_{ij} \geq 0, \; \forall\, i, j = 1, \ldots, n. \end{aligned}$$

(a) Show how a minimum mean-cycle can be recovered from an optimal flow f of the above LP.

(b) Provide the dual of this linear program.

Reserve space. Check the description on the first page for its use.

(a)

(b)