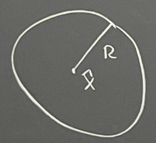
The ellipsoid method:

Def: $B(\vec{x}, R)$
$= \{x \in \mathbb{R}^n : \|x - \vec{x}\| \le R\}$



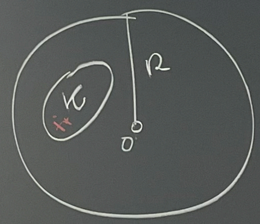
Ellipsoid method: What it does?

Solves: Given i) $R \in \mathbb{R}_{>0}$
ii) $\mu \in \mathbb{R}_+$, iii) $K \subseteq \mathbb{R}^n$

$K \subseteq B(0, R)$ convex

either find $x^* \in K$

or assert $vol(K) \le \mu$.



Proceeds in iterations

# of iterations

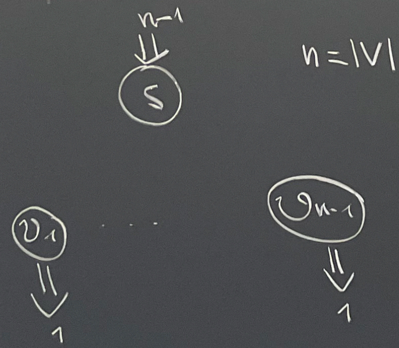
$$\boxed{O\left(\log\left(\frac{R}{\mu}\right) \cdot poly(n)\right)}$$

Definition.

Let $A \in \mathbb{R}^{n \times n}$ non-singular.
and $t \in \mathbb{R}^n$

The image of $B(0,1)$
under the map $f(x) = (Ax + t)$
is calle <u>ellipsoid</u> $\mathcal{E}(A, t)$

Model as Network Flow.



$n = |V|$

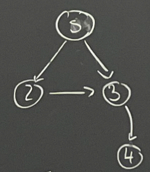
Min $c^T \cdot f$ (**)

$s \rightarrow A_G \, f = \begin{pmatrix} n-1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$

$f \geq 0$

$A_G(v,e) = \begin{cases} 1 & \text{if } v \text{ starting node of } e \\ -1 & \text{if } v \text{ is end-node of } e \\ 0 & \text{otherwise.} \end{cases}$

Example A_G:



$A_G = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

Thm: Let $G=(V,A)$ be a digraph. Then $A_G \in \{0,\pm1\}^{|V|\times|A|}$ is TU.

Proof. Let $A' \in \{0,\pm1\}^{k\times k}$ be a submatrix of A_G. We show $\det(A') \in \pm1, 0$ by induction on k.

$k=1$. $\det(A') = \pm1, 0$

$k>1$:

CASE 1. $\exists$ col with ≤ 1 nonzero entry.

$A' = \begin{pmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ & * & \\ & 0 & \\ & \vdots & \\ & 0 & \end{pmatrix}$ $* \in \pm1, 0$

develop.

$\det(A') = \{\pm1, 0\} * \det(A'')$

A'' is $(k-1)\times(k-1)$ submatrix of A_G

Induction $\Rightarrow \det(A') \in 0, \pm1$.

CASE 2: All columns of A' have two nonzero entries.

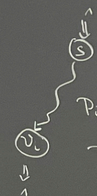
$\det(A') = 0$

$\left(\begin{array}{c|c|c} & 1 & \\ & \vdots & \\ & -1 & \end{array} \right) = \Sigma$

$\square$

Lemma: (**) is feasible $\Leftrightarrow \exists$ Path from s to each other vertex $v \in V$.

Proof. "$\Leftarrow$"



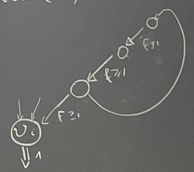
P_i simple path

$\chi(P_i) \in \{0,1\}^{|A|}$

$f = \sum_{i=1}^{n-1} \chi(P_i)$ feasible arc flow.

"⇒" let $f \in \mathbb{N}^{|A|}$ be o.p.l.

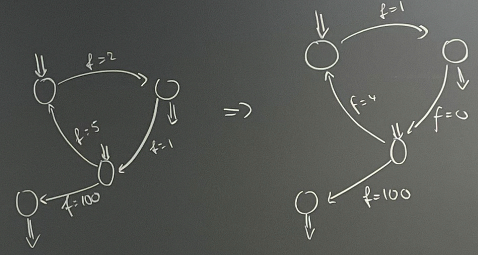
solution (**)



Backtracking does not create cycle, otherwise

reduce flow on arcs of cycle by 1

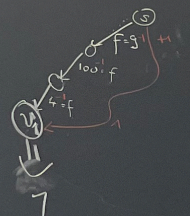
⇒ better feas. flow. ↯



<u>Theorem</u>: Let $G=(V,A)$, $s \in V$, $c: A \to \mathbb{R}^+$ be given and suppose $\exists$ path from s to every other vertex, then optimal integer flow is of the form $f = \chi(P_{v_1}) + \dots + \chi(P_{v_{n-1}})$

where P_{v_i} is shortest path from s to v_i

<u>proof</u>: Let f be optimal flow, $f \in \mathbb{N}^{|A|}$



Backtracking creates path P from s to v_1

P is of min cost, otherwise re-route one unit of flow

Along shorter path P'. yields feasible flow of smaller cost. ↯

f old f new.

$$c(f_{new}) = cost(f_{old}) - cost(P) + cost(P')$$