

König's Theorem.



Definition. Let $G=(V,E)$ be a graph. A subset $S \subseteq V$ is a vertex cover of G if

$$\forall e \in E. e \cap S \neq \emptyset$$

König's theorem.

Let $G=(V,E)$ be a bipartite graph. The max. cardinality of a matching of G equals the minimum card. of a vertex cover.

$M \subseteq E$ Matching
 $S \subseteq V$ vertex cover.

$$|M| \leq |S|$$

each $v \in S$
covers at most one $e \in M$

(P)

$$\max \sum_{e \in E} x(e)$$

$$\forall v \in V \sum_{e \in S(v)} x(e) \leq 1$$

$$x \geq 0$$

LP Relax for max-card.

Matching G bipartite

$\Rightarrow$ integral

(D) Variables: $y(v), v \in V$

$$\min \sum_{v \in V} y(v)$$

$$\forall v \in E: y(u) + y(v) \geq 1$$

$$y \geq 0$$

$$\min \sum_{v \in V} y(v)$$

$$A_G^T y \geq 1$$

$$y \geq 0$$

$$y \sim x(S)$$

$$S \subseteq V$$

(D) is LP-relaxation of min-card vertex cover.

proof (König's Thm)

A_G is TU.

(P) and (D) have opt O/L solutions corresponding to MATCHING AND VERTEX

Cover respectively.

DUALITY



$$A_G = \begin{bmatrix} v_1 & \dots & v_n \\ u_1 & \dots & u_m \end{bmatrix}$$

max $\sum x$

$$Ax \leq b$$

$$x \geq 0$$

min $\sum y$

$$A^T y \geq c$$

$$y \geq 0$$

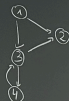
Network Flows

Def. A directed graph

is a tuple $G = (V, A)$

- V is finite set of vertices
- $A \subseteq V \times V$ is set of arcs or directed edges.

Example



$$V = \{1, 2, 3, 4\}$$

$$A = \{(1, 2), (1, 3), (3, 2), (3, 4), (4, 3)\}$$

Definition

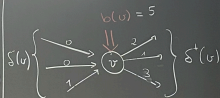
Let $G = (V, A)$ be a directed graph and $b \in \mathbb{R}^{|V|}$. A solution

$f \in \mathbb{R}_{\geq 0}^{|A|}$ of

$$\forall v: \sum_{e \in \delta^-(v)} f(e) - \sum_{e \in \delta^+(v)} f(e) = b(v)$$

$$(*) \quad f \geq 0$$

is called feasible arc-flow.



Minimum cost network flow

Given: - $G = (V, A)$

- $b \in \mathbb{R}^{|V|}$ (inflow)
- $c \in \mathbb{R}^{|A|}$ (cost)
- $u \in \mathbb{R}_{\geq 0}^{|A|}$ (capacities)

Task: Min $c^T f$ s.t. f is feasible flow

Necessary condition existence of feas. arc-flow

Suppose f sat. $(*)$ $\textcircled{u} \xrightarrow{\text{flow}} \textcircled{v}$

then: $\boxed{\sum_{v \in V} b(v) = 0}$

Necessary for existence of flow.

The shortest path problem:

Given: - $G = (V, A)$ digraph

- $C: A \rightarrow \mathbb{R}_{\geq 0}$ ARC-COSTS.
- $s \in V$ designated starting vertex.

Task: Compute shortest path distances from s to all other nodes $t \in V \setminus \{s\}$.

