The ellipsoid method.

Given: i) R, ii) $K \subseteq \mathbb{R}^n$, $K \subseteq B(0,R)$

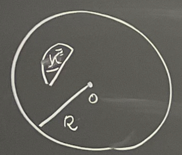
iii) $\mu \in \mathbb{R}_{>0}$

Either:

Finds $x^* \in K$

OR

ASSERTS $vol(K) < \mu$



Ellipsoid method proceeds in iterations. Number

$$\leq O\left(\log \frac{R}{\mu}\right) \cdot poly(n)$$

USE CASE 1: Given

$Ax \leq b$, $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $\|A\|_\infty \leq B$, $\|b\|_\infty \leq B$.

USE ELLIPSOID METHOD TO:

either. find $x^* \in \{Ax \leq b\}$

or assert $\{Ax \leq b\}$ is not full-dimensional.

$(vol\{Ax \leq b\} = 0)$

What should be R?

Suppose $x^* \in \{Ax \leq b\}$

add $x_i \geq 0$ if $x^*_i \geq 0 : I$

$x_i \leq 0$ if $x^*_i < 0 : J$

Now: $\{Ax \leq b,\ x_i \geq 0\ i \in I,\ x_j \leq 0\ j \in J\}$

is feasible and has vertices.

Therefore: Assume $\{Ax \leq b\}$ has vertices.

For basis $B \subseteq \{1, \dots, m\}$:

$$x^*_B = A_B^{-1} \cdot b_B$$

$$= \frac{\tilde{A}_B}{det(A_B)} \cdot b_B$$

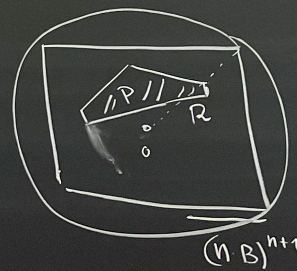
$$\|\tilde{A}_B\|_\infty \leq (\sqrt{n} \cdot B)^n$$

$$\leq (n \cdot B)^n$$

$$\|x^*_B\|_\infty \leq n \cdot (nB)^n \cdot B \leq (n \cdot B)^{n+1}$$

$\Rightarrow$ Add inequalities: to $Ax \leq b$

$$-(n \cdot B)^{n+1} \leq x \leq (n \cdot B)^{n+1}$$



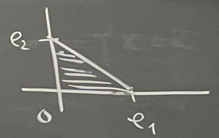
$$\boxed{R = \sqrt{n} \cdot (n \cdot B)^{n+1} \leq (n \cdot B)^{n+2}}$$

Task: Find $x^* \in P$ or assert P not full-dim.

Question: If $vol(P) > 0$ then find $\mu \in \mathbb{R}_{>0}$ s.th.

$$\boxed{vol(P) \geq \mu}$$

Exercise:

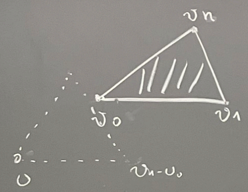


$$vol(\Sigma) = \frac{1}{n!}$$

where $\Sigma = conv\{0, e_1, \dots, e_n\}$

If P is full-dimensional, then $\exists$ vertices $v_0, v_1, \dots, v_n$ affinely independent

$\Leftrightarrow$ $v_1 - v_0, \dots, v_n - v_0$ lin-independent.



$$vol(conv(v_0, \dots, v_n))$$

$$= vol\left(conv\_hull\left(0, v_1 - v_0, \dots, v_n - v_0\right)\right)$$

$$= vol\left(\underbrace{(v_1 - v_0, \dots, v_n - v_0)}_{\substack{A' \in \mathbb{R}^{n \times n} \\ \text{non-singular.}}} \Sigma\right) = \frac{1}{n!} \left|det(A')\right|$$

Goal: lower-bound determinant.

$$\left| det\left(\overset{\substack{* \, det(A_{B_1}) \cdot det(A_{B_0}) \\ \downarrow}}{v_1 - v_0}, \dots, \underset{\substack{\uparrow \\ * \, det(A_{B_n}) \, det(A_{B_0})}}{v_n - v_0} \right) \right|$$

$$v_i = \frac{\tilde{v}_i \in \mathbb{Z}^n}{\left|det(A_{B_i})\right|}$$

$$\Rightarrow \left| det\left(v_1 - v_0, \dots, v_n - v_0\right) \right|$$

$$\geq \frac{1}{\prod_{i=1}^{n} \underbrace{det(A_{B_i}) \cdot det(A_{B_0})}_{\leq (\sqrt{n} \cdot B)^n}}$$

$$\geq \frac{1}{(n \cdot B)^{2n^2}}$$

$$\Rightarrow \mu \geq \frac{1}{(n \cdot B)^{2n^2}} \quad \frac{1}{n!} \geq \frac{1}{n^n}$$

$$\geq \frac{1}{(n \cdot B)^{3n^2}}$$

Concequently:

Given $\{Ax \leq b\}$, $\|A\|_\infty \leq B \in \mathbb{N}$

$\|b\|_\infty \leq B$

After: $O\left(\log\left((n \cdot B)^{n+2} \cdot (n \cdot B)^{3n^2}\right)\right)$

$\cdot$ poly(n) iterations:

We found feasible x^* or can

assert that

$\{Ax \leq b\}$ is not full

dimensional.

$$\boxed{O(n^2 \cdot \text{poly}(n) \cdot \log(n \cdot B))}$$

Recall: $\log(B) \sim$ # of bits

to encode $B \in \mathbb{N}$

The ellipsoid method.

Given: i) R, ii) $K \subseteq \mathbb{R}^n$, $K \subseteq B(0,R)$

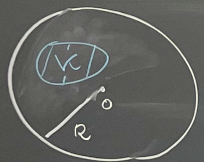
iii) $\mu \in \mathbb{R}_{>0}$, $\mu \leq 1$

Either:

Finds $x^* \in K$

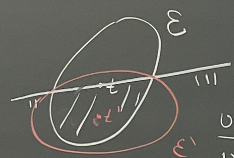
OR

ASSERTS $vol(K) < \mu$



Ellipsoid method proceeds in iterations. Number

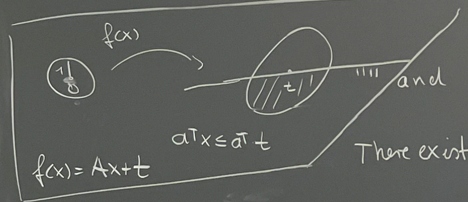
$\leq O\left(\log \frac{R}{\mu}\right) \cdot poly(n)$



$\frac{vol(\mathcal{E}')}{vol(\mathcal{E})} \leq \left(\frac{1}{e}\right)^{\frac{1}{2(n+1)}} < 1$

Lemma. Half ellipsoid lemma:

Let $\mathcal{E}(A,t) \subseteq \mathbb{R}^n$ be an ellipsoid, and $a \in \mathbb{R}^n \setminus \{0\}$.



There exists an ellipsoid $\mathcal{E}'$ containing

$\mathcal{E}(A,t) \cap (a^T x \leq a^T t)$ s.t. $\frac{vol(\mathcal{E}')}{vol(\mathcal{E})} \leq \left(\frac{1}{e}\right)^{\frac{1}{2(n+1)}}$

The method works with

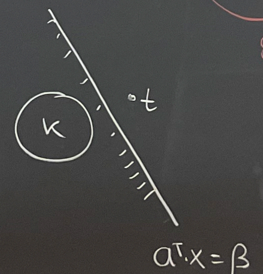
$\mathcal{E}$ ellipsoid s.th. $\boxed{\mathcal{E} \supseteq K \text{ (INVARIANT)}}$

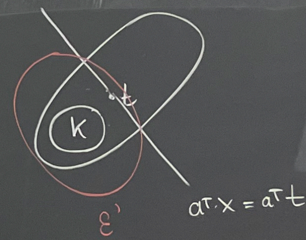
$\mathcal{E} := B(0,R)$ initial ellipsoid

INVARIANT is TRUE.

WHILE $vol(\mathcal{E}) > \mu$

- check $t \in K$
  - t: center of $\mathcal{E}$
  - if $t \in K$: return t.
- otherwise let $a^T x = \beta$ be separating hyperplane (t and K)
- Replace: $\mathcal{E} := \mathcal{E}'$





Number of iterations until

$\operatorname{vol}(\mathcal{E}) < \mu$

ellipsoid after iteration i

$$\mu \le \operatorname{vol}\left(\mathcal{E}^{(i)}\right)$$

$$\le \left(\frac{1}{e}\right)^{\frac{i}{2(n+1)}} \cdot \operatorname{vol}\left(\mathcal{E}_{init}\right)$$

$$= \left(\frac{1}{e}\right)^{\frac{i}{2(n+1)}} \cdot \underbrace{\operatorname{vol}(V_n)}_{\le 1} \cdot R^n$$

$$\le \left(\frac{1}{e}\right)^{\frac{i}{2(n+1)}} \cdot R^n$$

$$\iff e^{\frac{i}{2(n+1)}} \le \frac{R^n}{\mu}$$

$$\iff \frac{i}{2(n+1)} \le \ln\left(\frac{R^n}{\mu}\right) \qquad \boxed{\mu \le 1}$$

$$\iff i \le 2(n+1)\ln\left(\frac{R^n}{\mu^n}\right)$$

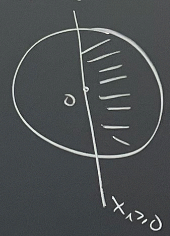
$$= 2(n+1)\cdot n \cdot \ln\left(\frac{R}{\mu}\right)$$

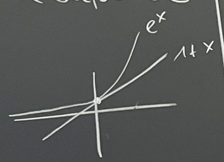
If $\mu \le 1$, the number of iterations

$$\le O(n^2)\cdot \ln\left(\frac{R}{\mu}\right)$$

$$\mathcal{E}(A,b) = \left\{x \in \mathbb{R}^n \;\middle|\; \|A^{-1}(x-b)\|_2 \le 1\right\}$$

Proof of Half-Ellipsoid Lemma:





$$1 + x \le e^x$$

<u>Half-Ball:</u> $\{x \in \mathbb{R}^n : \|x\| \le 1,\ x_1 \ge 0\} := H$

$$H \subseteq \left\{x \in \mathbb{R}^n : \left(\frac{n+1}{n}\right)^2 \cdot \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}\sum_{i=2}^{n} x_i^2 \le 1\right\}$$

$$A^{-1} = \begin{bmatrix} \frac{n+1}{n} & & & 0 \\ & \sqrt{\frac{n^2-1}{n^2}} & & \\ & & \ddots & \\ 0 & & & \sqrt{\frac{n^2-1}{n^2}} \end{bmatrix} \qquad = e^{-\frac{1}{2(n+1)}}$$

$$\det(A) = \underbrace{\frac{n}{n+1}}_{\left(1-\frac{1}{n+1}\right)} \cdot \underbrace{\left(\frac{n^2}{n^2-1}\right)}_{1+\frac{1}{n^2-1}}^{\frac{n-1}{2}} \le e^{-\frac{1}{n+1}} \cdot \left(e^{\frac{1}{n^2-1}}\right)^{\frac{n-1}{2}} = e^{-\frac{2}{2(n+1)}} \cdot e^{\frac{1}{2(n+1)}}$$