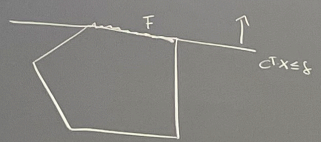


$F \sim \quad c^T \cdot x = \delta \quad x \in \mathbb{R}^n$

$Ax \leq b$

$\text{rank}\begin{pmatrix} c^T \\ -c^T \\ A \end{pmatrix} = n$

$\Rightarrow F \neq \emptyset \Rightarrow F$ has vertex

Recap: Integer Programming.

$\text{MAX } c^T \cdot x$

$Ax \leq b$

$x \in \mathbb{Z}^n \leftarrow x \in \mathbb{R}^n$.

linear Program

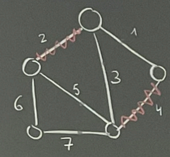
Allows to model decisions

IP is NP-hard

$G = (V, E)$

Matching: $M \subseteq E$

$\forall e_1 \neq e_2 \in M.\ e_1 \cap e_2 = \emptyset$



$M \sim \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

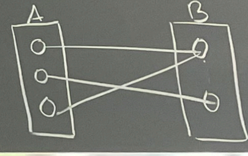
Characteristic vector

$\chi(M) \in \{0,1\}^{|E|}$

M. Matching of max. cardinality.

G is bipartite. If $\exists$ partition $V = A \dot{\cup} B$ (DISJOINT)

s.th. $\forall e \in E.\ |e \cap A| = 1$



$\leftarrow$ Example

The problem of finding a max cardinality matching

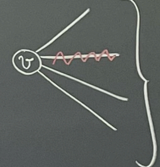
If G is bipartite, then Simplex-Alg. will find one.

Model max. card. matching as IP.

Given $G = (V, E)$

Variables: $x(e) = \begin{cases} 1 & \text{if } e \text{ is in } M \\ 0 & \text{otherwise} \end{cases}$

$\delta(v) = \{e \in E : v \in e\}$



IP:

$$\text{MAX} \sum_{e \in E} x(e)$$

$$\forall v \in V: \sum_{e \in \delta(v)} x(e) \leq 1$$

$$x \geq 0$$

$$x \in \mathbb{Z}^{|E|}$$

$x \in \{0,1\}^{|E|}$

IMPORTANT PRINCIPLE.

Linear Programming relaxation

$$\text{Max} \sum_{e \in E} x(e)$$

$$\forall v \in V: \sum_{e \in \delta(v)} x(e) \leq 1$$

(LP)

$$x \geq 0$$

$$x \in \mathbb{R}^{|E|}$$

OBSERVATION:

LP $\geq$ IP (I)

If LP has optimal solution $x^* \in \mathbb{Z}^{|E|}$,

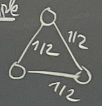
then x^* is IP-feasible

(I) $\Rightarrow x^*$ is opt. sol of IP.

It turns out:

If G is bipartite, then every basic feasible solution of LP is integral. $\Rightarrow$ Simplex can be applied to find max-cardinality matching

Example



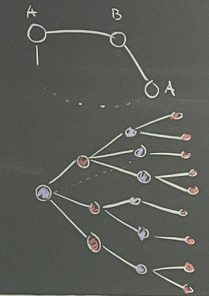
$LP = 3/2$

$IP = 1$

Exercise: $G=(V,E)$ is bipartite

$\Leftrightarrow$ G does not have a cycle containing odd number of vertices

"$\Rightarrow$"



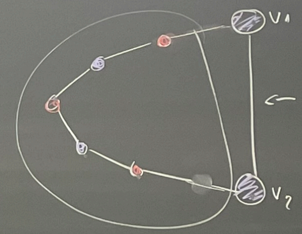
Cycle is even

"$\Leftarrow$"

A
B

Spanning tree.

take arbitrary edge and suppose both end points are blue



$\leftarrow$ not in tree

Unique path in tree connecting v_1, v_2

odd cycle ↯.

To show: each basic feasible solution of (LP) has integer components only!

Lemma: Let $A \in \mathbb{Z}^{n\times n}$ non-singular. The system $Ax=b$ has integer solution for each $b \in \mathbb{Z}^n$

$\Leftrightarrow$ $\det(A) = \pm 1$

proof: "$\Rightarrow$" $A x^{(i)} = e_i$ $i=1,\dots,n$

has integer solution $x^{(i)} \in \mathbb{Z}^n$.

$$A\underbrace{(x^{(1)}, x^{(2)}, \dots, x^{(n)})}_{A^{-1}} = I_n$$

$$\Rightarrow \underbrace{\det(A)}_{\in\mathbb{Z}}\ \underbrace{\det(A^{-1})}_{\in\mathbb{Z}} = 1$$

$$\Rightarrow \det(A) = \pm 1$$

"$\Leftarrow$"

$$A^{-1} = \frac{\tilde{A}}{\det(A)} \in \mathbb{Z}^{n\times n} \quad \boxtimes$$

$$\det(A) = \sum_{j=1}^{n} a_{ij}\cdot(-1)^{i+j}\cdot\det(A_{ij})$$

$$\begin{array}{cc} & j \\ i & \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array} \end{array}$$

delete.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn}\end{bmatrix} \cdot \underbrace{\begin{bmatrix} \det(A_{11}) & \det(A_{21})(-1)^{2+1} & \cdots & \det(A_{n1})(-1)^{n+1} \\ (-1)^{1+2}\det(A_{12}) & \vdots & & \vdots \\ (-1)^{1+n}\det(A_{1n}) & & & \det(A_{nn})(-1)^{n+n}\end{bmatrix}}_{\tilde{A}}$$

$$= \det(A)\cdot I_n$$

P:

$$\begin{pmatrix} A \\ -I \end{pmatrix} x \le \begin{bmatrix} b \\ 0 \end{bmatrix} \qquad \boxed{A} x \le b, \quad A \in \mathbb{Z}^{m \times n}$$

$$x \ge 0 \qquad b \in \mathbb{Z}^m$$

- P has vertices (if $\neq \emptyset$)
- Is there a condition that we can impose on A s.th. all vertices are integral
- Is this condition satisfied by bipartite Matching

Let $B \subseteq \{1, \dots, m+n\}$ be a basis, $|B| = n$

Split $B = B_1 \,\dot\cup\, B_2$, where

$B_1 \subseteq \{1, \dots, m\}$, $B_2 \subseteq \{m+1, \dots, m+n\}$

$B_2 = \{m+i_1, \dots, m+i_k\}$

$$A_B = \begin{bmatrix} A_{B_1} \\ \hline -e_{i_1}^T \\ \vdots \\ -e_{i_k}^T \end{bmatrix}$$

$$A_B = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

WOULD LIKE: $\det(A_B) = \pm 1$

Since then $x^*_B = A_B^{-1} \cdot b_B \in \boxed{\mathbb{Z}^n}$

$$\det(A_B) = (\pm 1) \cdot \det(\tilde{A}_{B_1})$$

$A_{\tilde{B}_1}$ stems from A_{B_1} by deletion of columns $i_1, \dots, i_k$.

This $A_{\tilde{B}_1}$ is a $(n-k) \times (n-k)$ submatrix of A

[choose $n-k$ rows and $n-k$ columns of A]

Definition. $A \in \mathbb{Z}^{m \times n}$ is called totally unimodular (TU) if $\forall k \in \{1, \dots, n\}$ each $k \times k$ submatrix of A has determinant $\pm 1, 0$

We proved:

Theorem. Let $A \in \mathbb{Z}^{m \times n}$ be TU. Then $P = \{x \in \mathbb{R}^n : Ax \le b, x \geq 0\}$ is integral (integer vertices only) for each $b \in \mathbb{Z}^m$.

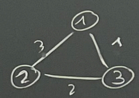
Theorem: (Hoffman & Kruskal)
$A \in \mathbb{Z}^{m \times n}$ is TU $\iff \forall b \in \mathbb{Z}^m$ $P = \{x \in \mathbb{R}^n : Ax \le b, x \geq 0\}$ is integral.

$$\max \sum_{e \in E} x(e) \qquad \text{MATCHING LP.}$$

$$\boxed{v \in V: \quad \sum_{e \in \delta(v)} x(e) \le 1}$$

$$\sim A_G \qquad x \ge 0$$

Example:



$$A_G = \begin{array}{c} \\ \\ v_3 \to \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$
$$\qquad\qquad\quad \uparrow$$
$$\qquad\qquad\quad e_2$$

Definition. Let $G = (V, E)$ be a graph. $\boxed{A_G} \in \{0,1\}^{|V| \times |E|}$ is node-edge incidence matrix of G

With: $(A_G)_{v,e} = \begin{cases} 1 & \text{if } v \in E \\ 0 & \text{otherwise} \end{cases}$

Matching

$$\max \sum_{e \in E} x(e)$$

$$A_G x \leq \mathbb{1}$$

$$x \geq 0$$

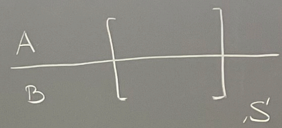
$$x \in \mathbb{Z}^n$$

Theorem: Let $G = (A+B, E)$ be a bipartite graph. A_G is totally unimodular.

proof: Need to show: each $k \times k$ submatrix has determinant $\pm 1, 0$

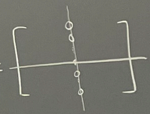
INDUCTION ON k: $k = 1$ (✓)

$k > 1$ $S \in \{0,1\}^{k \times k}$ be sub-matrix of A_G.



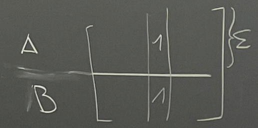
CASE 1: S has 0-column $\Rightarrow \det(S) = 0$

CASE 2: S has a column with exactly one "1".

$S =$  $\det(S) = \pm 1 \det(\tilde{S})$

$\in \pm 1, 0$ by IND.

CASE 3: All columns of S have 2 ones.

$S =$ 

$\det(S) = 0$ ∎