# Review: Duality

$a_1^T x^* = b_1$

$a_2^T \cdot x^* = b_2$

MAX $c^T \cdot x$ (P)

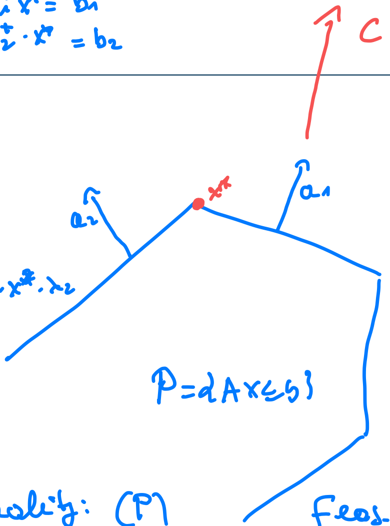
$Ax \le b$

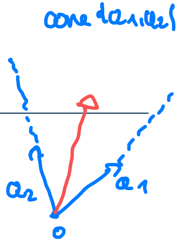
$b^T \cdot \lambda = b_1 \cdot \lambda_1 + b_2 \cdot \lambda_2$

$= a_1^T \cdot x^* \cdot \lambda_1 + a_2^T \cdot x^* \cdot \lambda_2$

$= c^T \cdot x^*$

$\lambda^T \cdot A = c^T$




$\exists\ \lambda_1, \lambda_2 \ge 0$

s.th.

$c = \lambda_1 a_1 + \lambda_2 \cdot a_2$

**DUAL:** min $b^T \cdot \lambda$

$\lambda^T A = c^T$

$\lambda \ge 0$

**Duality:** (P) Feas. & bounded

then (D) is feas & bounded

BOTH OPTIMAL VALUES coincide.

# Dual of the Dual

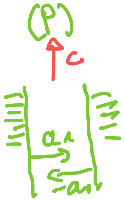
**Corollary**

*If the dual linear program has an optimal solution, then so does the primal linear program and the objective values coincide.*

# Table of possibilities

| P \ D | infeasible | unbounded | bounded & feas. |
|---|---|---|---|
| infeasible | possible | possible | No |
| unbounded | possible | No | No |
| bounded & feas | No | No | possible. |

$c \perp$ all NORMAL VECTORS



# Further example

Derive $c^T \cdot x \leq \beta$ valid for each

$x^*$ feasible (P)

$$
\begin{array}{rrcl}
 & \max & c^T x & \\
\lambda_1 & \equiv Bx & = & b \\
\lambda_2 \geq 0 & \equiv Cx & \leq & d.
\end{array}
$$

(Primal)

$\min \lambda_1^T \cdot b + \lambda_2^T \cdot d$

$\lambda_1^T \cdot B + \lambda_2^T \cdot C = C$

$\lambda_2 \geq 0$

$$
\begin{array}{rcl}
\min b^T y_1 + d^T y_2 & & \\
B^T y_1 + C^T y_2 & = & c \\
y_2 & \geqslant & 0.
\end{array}
$$

(Dual)

$$\max \; C^T . x$$
$$Bx = b$$
$$Cx \leq d$$

$\leq$ Standard FORM. $\Longrightarrow$

$$\text{MAX} \; c^T . x$$
$$Bx \leq b$$
$$-Bx \leq -b$$
$$Cx \leq d$$

$$\sim \quad \max \; c^T . x \qquad \begin{bmatrix} B \\ -B \\ C \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \\ d \end{bmatrix} \begin{matrix} \} \lambda_1 \\ \} \lambda_2 \\ \} \lambda_3 \end{matrix}$$

$$\lambda_4 = \lambda_1 - \lambda_2$$

**Dual:**

$$\text{Min} \begin{bmatrix} b \\ -b \\ d \end{bmatrix}^T \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

$$B^T . \lambda_1 - B^T . \lambda_2 + C^T . \lambda_3 = C \quad \sim$$

$$\lambda_1, \lambda_2 \geq 0 \qquad \lambda_3 \geq 0$$

B has $m_1$ rows

$$\lambda_1, \lambda_2 \in \mathbb{R}^{m_1}$$

$$\lambda_3 \in \mathbb{R}^{m_2}$$

C has $m_2$ rows.

$$\min \; b^T . \lambda_4 + d^T . \lambda_3$$
$$B^T . \lambda_4 + C^T \lambda_3 = C$$
$$\lambda_3 \geq 0$$

Re-WRITE The above.

$$\min \; b^T . \lambda_1 + d^T . \lambda_2$$
$$B^T . \lambda_1 + C^T . \lambda_2 = C$$
$$\lambda_2 \geq 0$$

# Zero sum games

$\max_i \min_j a_{ij} = 2$

$\min_j \max_i a_{ij} = 2$

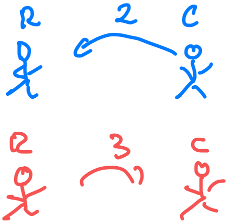
Robert

Carla.

$$A = \begin{pmatrix} 5 & 1 & 3 \\ 3 & 2 & 4 \\ -3 & 0 & 1 \end{pmatrix}$$

Row player: Chooses row $i$ 2, | 3

Column player: Chooses column $j$ 2 | 1



# Rock – paper – scissors

| | | | |
|---|---|---|---|
| | 0 | -1 | 1 |
| | 1 | 0 | -1 |
| | -1 | 1 | 0 |

# Deterministic strategies

Robert begins

$$\max_i \min_j \; a_{ij}$$

Carla begins

$$\begin{matrix} \min & \text{MAX} \\ \cancel{\max_j} & \cancel{\min_i} \end{matrix} \; a_{ij}$$

Example:

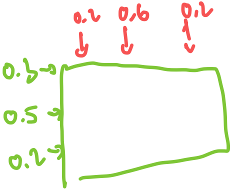
$$\begin{pmatrix} 1.2 & -3 \\ -2 & 1 \end{pmatrix}$$

$$\left.\begin{aligned} &\max_i \overbrace{\left(\min_j a_{ij}\right)}^{f(i)} = -2 \\ &\min_j \underbrace{\max_i a_{ij}}_{g(j)} = 1 \end{aligned}\right\} \neq$$

# Mixed strategies

## Definition

Let $A \in \mathbb{R}^{m \times n}$ define a two-player matrix game. A mixed strategy for the row-player is a vector $x \in \mathbb{R}^m_{\geqslant 0}$ with $\sum_{i=1}^m x_i = 1$. A mixed strategy for the column player is a vector $y \in \mathbb{R}^n_{\geqslant 0}$ with $\sum_{j=1}^n y_{\cancel{i}\,j} = 1$.




$$E[\text{Payoff}] = x^T A y. \tag{5}$$

$= \sum_{ij} a_{ij} \cdot P(\text{r chooses } j \text{ \& c chooses } i)$

independent

$= x_i \cdot y_j$

# Example: Mixed strategy rock – paper – scissors

$$R = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} \qquad C = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$E[\text{PAYOFF}] = 0$$

$$\max_x \min_y E[\text{PAYOFF}] = 0$$

$$\min_y \max_x E[\text{PAYOFF}] = 0$$

# Weak duality

$x' \in X$ where $\max_{x \in X} (f(x))$ is attained

$f(x) = \min_{y \in Y} x^T \cdot A \cdot y.$

Lemma

*Let $A \in \mathbb{R}^{m \times n}$, then*

$$\max_{x \in X} \min_{y \in Y} x^T A y \leq \min_{y \in Y} \max_{x \in X} x^T A y,$$

*where $X$ and $Y$ denote the set of mixed row and column-strategies respectively.*

proof: Let $x', y'$ be fixed mixed stategies of R and C respectively

$$\min_{y \in Y} x' \cdot A \cdot y \leq x' \cdot A \cdot y' \leq \max_{x \in X} x A \cdot y'$$

$\square$

# Minimax-Theorem

**Theorem (von Neumann (1928))**

$$\max_{x \in X} \min_{y \in Y} x^T A y = \min_{y \in Y} \max_{x \in X} x^T A y,$$

*where $X$ and $Y$ denote the set of mixed row and column-strategies respectively.*

Proof: Based on lin. prog. Duality

Row Player First:

$$\max_{x \in X} \boxed{\min_{y \in Y} \underbrace{x^T \cdot A}_{\text{obj.}} \cdot y}$$

here: x is fixed,
Linear Program.

Duality.

$$\min \; x^T \cdot A \cdot y$$
$$\sum_{j=1}^{n} y_j = 1$$
$$y \geq 0$$

$\Downarrow$ $=$

$$\max \; x_0$$
$$\mathbb{1}^T \cdot x_0 - A^T \cdot x \leq 0$$

$\sim$ $\max \; x_0$

$$\mathbb{1} \, x_0 - A^T \cdot x \leq 0 \quad \Big| \; \begin{matrix} y_1 \\ \vdots \\ y_m \end{matrix} \geq 0$$
$$\sum_{i=1}^{n} x_i = 1 \quad | \; y_0$$
$$x \geq 0 \quad | \; y'$$
$$-I \cdot x \leq 0$$

DUAL $\Longrightarrow$

$$\boxed{\begin{aligned} &\min \; y_0 \\ &\sum_{j=1}^{m} y_j = 1 \\ &\mathbb{1} \cdot y_0 - A \cdot y \geq 0 \\ &y \geq 0 \end{aligned}}$$

## Column player first.

$$\min_{y \in Y} \underbrace{\max_{x \in X} x^T \cdot A \cdot y}$$

$\min y_0$

$\mathbb{1} \cdot y_0 - A \cdot y \geq 0$

$\sum y_j = 1$

$y \geq 0$

$\min y_0$

$\mathbb{1} \cdot y_0 - A \cdot y \geq 0$

$\sum_{j=1}^{n} y_j = 1$

$y \geq 0$

$\Rightarrow$ MAX MIN = MIN MAX $\square$

$$\max \; x_0$$
$$\mathbb{1}\, x_0 - A^T \cdot x \le 0 \quad \Big| \; \begin{matrix} y_1 \\ \vdots \\ y_m \end{matrix} \ge 0$$
$$\sum_{i=1}^{n} x_i = 1 \quad \big| \; y_0$$
$$-I x \le 0 \quad \big| \; y'$$

DUAL $\Longrightarrow$

$$\boxed{\begin{aligned} &\min \; y_0 \\ &\sum_{j=1}^{m} y_j = 1 \\ &\mathbb{1} \cdot y_0 - A \cdot y \ge 0 \\ &y \ge 0 \end{aligned}}$$

HOW To derive this in greater detail.

Recall:

(P)

$$\max \; c^T \cdot x$$
$$Bx = b$$
$$Cx \le d$$
$$y \in \mathbb{R}^n$$

(D)

$$\min \; b^T \cdot y_1 + d^T \cdot y_2$$
$$B^T \cdot y_1 + C^T \cdot y_2 = c$$
$$y \ge 0$$

$$C = \begin{bmatrix} 1 & \\ \vdots & -A^T \\ 1 & \\ 0 & \\ \vdots & -I \\ 0 & \end{bmatrix}$$

$$B = (1, \dots, 1)$$

$$d = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$b = (1)$$

DUAL

$$\min \; y_0 + \cancel{0^T \cdot y} + \cancel{0^T \cdot y'}$$

$$\boxed{\begin{aligned} &\begin{pmatrix} 0 \\ \mathbb{1} \end{pmatrix} y_0 + \begin{pmatrix} \mathbb{1}^T & 0^T \\ -A & -I \end{pmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &y, y' \ge 0 \end{aligned}}$$

$\iff$

$$\sum y_j = 1$$
$$\mathbb{1} \cdot y_0 - A \cdot y \ge 0$$
$$y \ge 0$$

# Duality via Farkas' lemma

**Theorem (Second variant of Farkas' lemma)**

*Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. The system $Ax \leq b$ has a solution if and only if for all $\lambda \geqslant 0$ with $\lambda^T A = 0$ one has $\lambda^T b \geqslant 0$.*

# Duality via Farkas' lemma

# Algorithms and running time analysis

Consider the following algorithm to compute the product of two $n \times n$ matrices $A, B \in \mathbb{Q}^{n \times n}$:

**for** $i = 1, \ldots, n$
    **for** $j = 1, \ldots, n$
        $c_{ij} := 0$
        **for** $k = 1, \ldots, n$
            $c_{ij} := c_{ij} + a_{ik} \cdot a_{kj}$

# $O$-notation

**Definition**

Let $T, f : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ be functions. We say

- $T(n) = O(f(n))$, if there exist positive constants $n_o \in \mathbb{N}$ and $c \in \mathbb{R}_{>0}$ with
$$T(n) \leq c \cdot f(n) \text{ for all } n \geqslant n_0.$$
- $T(n) = \Omega(f(n))$, if there exist constants $n_o \in \mathbb{N}$ and $c \in \mathbb{R}_{>0}$ with
$$T(n) \geqslant c \cdot f(n) \text{ for all } n \geqslant n_0.$$
- $T(n) = \Theta(f(n))$ if
$$T(n) = O(f(n)) \text{ and } T(n) = \Omega(f(n)).$$

# Example

**Example**

The function $T(n) = 2n^2 + 3n + 1$ is in $O(n^2)$, since for all $n \geqslant 1$ one has $2n^2 + 3n + 1 \leq 6n^2$. Here $n_0 = 1$ and $c = 6$. Similarly $T(n) = \Omega(n^2)$, since for each $n \geqslant 1$ one has $2n^2 + 3n + 1 \geqslant n^2$. Thus $T(n)$ is in $\Theta(n^2)$.

# Efficient algorithm, first definition

An algorithm runs in polynomial time, if there exists a constant $k$ such that the algorithm runs in time $O(n^k)$, where $n$ is the length of the input of the algorithm.