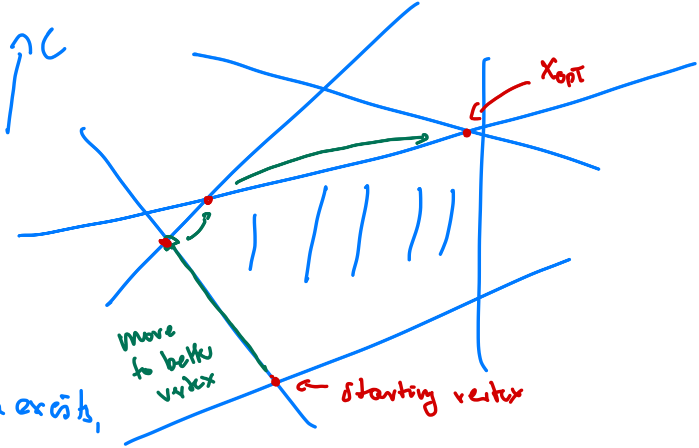
The simplex algorithm

$$\max c^T x$$
$$Ax \leq b$$

$A \in \mathbb{R}^{m \times n}$

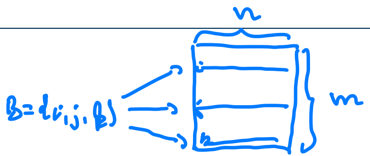
$\mathrm{rank}(A) = n$

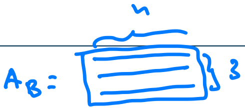


know: If opt. solution exists,

then $\exists$ optimal solution that is vertex

Notation





Let $B \subseteq \{1, \ldots, m\}$

$A_B \in \mathbb{R}^{|B| \times n}$ rows indexed by B

$b_B \in \mathbb{R}^{|B|}$ components indexed by B

Example: For $A = \begin{pmatrix} 3 & 2 \\ 7 & 1 \\ 8 & 4 \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$

and $B = \{2, 3\}$, one has

$$A_B = \begin{pmatrix} 7 & 1 \\ 8 & 4 \end{pmatrix} \text{ and } b_B = \begin{pmatrix} 2 \\ 6 \end{pmatrix}.$$

$\max c^T x$

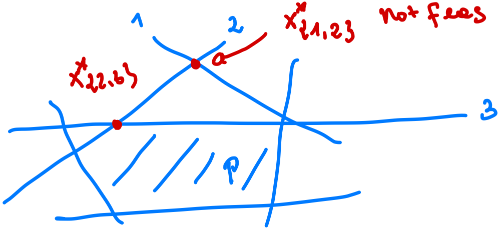
$Ax \leq b$

Feasible basis

Rows of A_B are basis of $\mathbb{R}^n$

Definition

An index set $B \subseteq \{1, \ldots, m\}$ is a basis if $|B| = n$ and A_B is non-singular. If in addition $x^*_B = A_B^{-1} b_B$ is feasible, then B is called a feasible basis.



$\{2,3\}$ is feasible basis.

Feasible basis vs extreme point

if $P = \{Ax \le b\}$

remember: $v \in P$ is vertex $\Leftrightarrow$ subsystem $A'x \le b' < Ax \le b$ s.t. $A' \cdot v = b'$ satisfies $\text{rank}(A') = n$

A

n lin. indep rows $\Rightarrow B$

$A' \cdot x^* = b'$

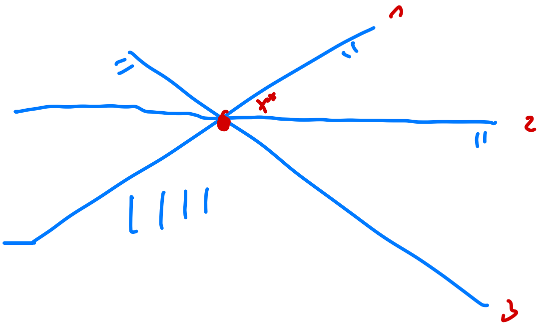
If $x^* \in P$ is vertex, then

$x^* = x^*_B$ for some feasible basis B

$= A_B^{-1} \cdot b_B$

if B is feasible basis $\Rightarrow$ x^*_B is vertex

Feasible basis vs extreme point



x^* is vertex represented by feasible basis

$\{1,2\}$, $\{2,3\}$ and $\{1,3\}$

(degenerate)

Optimal basis

$\max \; c^T \cdot x$
$Ax \leq b$

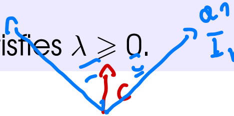
Definition

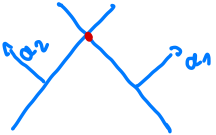
A basis B is called optimal if it is feasible and the unique $\lambda \in \mathbb{R}^m$ with

$$\lambda^T A = c^T \quad \text{and } \lambda_i = 0, \; i \notin B \tag{3}$$

satisfies $\lambda \geq 0$.

In other words if B is feasible and unique $\lambda_B \in \mathbb{R}^n$ with $\lambda_B^T \cdot A_B = c^T$ satisfies $\lambda_B \geq 0$





$c^T = \lambda_1 a_1 + \lambda_2 \cdot a_2 \quad \lambda_1, \lambda_2 \geq 0$

Optimal basis vs. optimal solution

Theorem

If B is an optimal basis, then $x^*_B = A_B^{-1} b_B$ is an optimal solution of the linear program. $\max c^T \cdot x$

$Ax \le b$

To show: x^*_B is opt. sol. of $LP(2)$

proof: $\max c^T \cdot x$ $\le$ $\max c^T \cdot x$

$Ax \le b$ $\quad$ $A_B x \le b_B$

$LP(1)$ $\quad$ $LP(2)$

Let $z \in \mathbb{R}^n$ be feasible for $LP(2)$

$c^T \cdot z = \underbrace{\lambda_B^T}_{\ge 0} \cdot \underbrace{A_B \cdot z}_{\le b_B}$

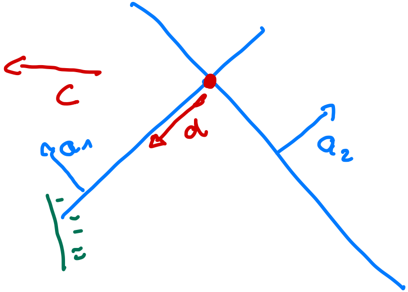
$\le \lambda_B^T \cdot b_B = \underbrace{\lambda_B^T A_B}_{= c^T} \cdot x^*_B$

Moving to an improving vertex

$i \uparrow \; \downarrow j \quad B \rightsquigarrow B'$

$B \subseteq \{1,\dots,m\}$ feas. basis.

not optimal i.e. $\exists\, i \in B \quad \lambda_i < 0$ for $\lambda_B^T \cdot A_B = c^T$



$d \in \mathbb{R}^n$ unique solution to

$$a_j^T d = \begin{cases} 0 & \text{for } j \in B \setminus \{i\} \\ -1 & \text{if } j = i. \end{cases} \qquad c^T d =$$

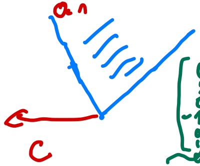
$\lambda_B^T \cdot A_B = c^T$

Example: d: $d \perp a_1$ $(d^T \cdot a_1 = 0)$

$d^T \cdot a_2 = -1$

$\lambda_1 < 0$

$\lambda_2 < 0$



$$\begin{array}{cc} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{c} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \end{array} \end{array}$$

Why improving: $c^T(x_B^* + \varepsilon \cdot d) = c^T x_B^* + \varepsilon \underbrace{c^T d}_{>0}$

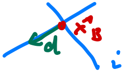
$c^T \cdot d = \lambda_B^T A_B \cdot d > 0$

How far can we move?

$B \xrightarrow{} i$, $B \xrightarrow{} j$

$x^*_B = A_B^{-1} \cdot b_B$

$$A_B \cdot d = \begin{bmatrix} 0 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$$



$$A\left[x^*_B + \varepsilon \cdot d\right] \le b$$

$$\underbrace{A x^*_B}_{\le b} + \varepsilon \cdot A \cdot d \le b$$

$K = \{k : (A \cdot d)_k > 0\}$

Danger $K \subseteq \{1, \dots, m\}$

If $K = \emptyset \Rightarrow$ LP unbounded.

WANT: $\varepsilon \cdot Ad \le b - A \cdot x^*_B$ $\quad \varepsilon > 0$

$\Leftrightarrow \forall k \in K.\ \varepsilon \cdot (A \cdot d)_k \le (b - A \cdot x^*_B)_k$

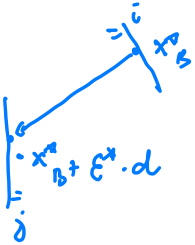
$\Leftrightarrow \forall k \in K \quad \varepsilon \le \dfrac{(b - A \cdot x^*_B)_k}{(A\, d)_k}.$

Otherwise let $j \in K$ be index where min $\min_{k \in K} \dfrac{(b - A \cdot x^*_B)_k}{(A \cdot d)_k}$ is attained. $B' = B - i + j$

How far can we move?

B' is feasible basis: for $\varepsilon^* = \min = \dfrac{(b - A \cdot x_B^*)_j}{(A \cdot d)_j}$

one has: $A_{B'}(x_B^* + \varepsilon^* \cdot d) = b_{B'}$

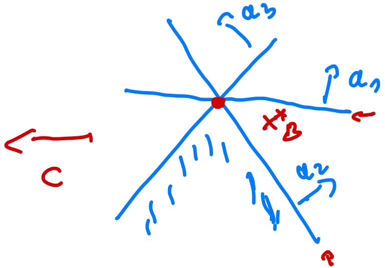


B' basis: $d \perp B \setminus \{i\}$.

$d \not\perp a_j \Rightarrow$ Rows indexed by B' are lin indep. $\Rightarrow$ B' is feasible basis

By construction: $c^T(x_B^* + \varepsilon^* \cdot d) \geq c^T(x_B^*)$.

How far can we move?



$B = \{1, 2\}$.

Suppose 2 lines

$\mathbb{R}^2$

$$A \cdot x^*_B - b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The simplex algorithm

Start with feasible basis B

`while` B is not optimal ($\exists i \in B: \lambda_i < 0$ with $\lambda_B^T \cdot A_B = c^T$)

Let $i \in B$ be index with $\lambda_i < 0$
Compute $d \in \mathbb{R}^n$ with $a_j^T d = 0$, $j \in B \setminus \{i\}$ and $a_i^T d = -1$
Determine $K = \{k \colon 1 \leq k \leq m,\ a_k^T d > 0\}$

`if` $K = \emptyset$
  assert LP unbounded

`else`
  Let $k \in K$ index where $\min\limits_{k \in K}(b_k - a_k^T x^*)/a_k^T d$ is attained
  update $B := B \setminus \{i\} \cup \{k\}$

```python
from sympy import *
A = Matrix([[1, 2, 2],
            [2, 1, 2],
            [2, 2, 1],
            [-1, 0, 0],
            [0, -1, 0],
            [0, 0, -1]])

b = Matrix([10,14,11,0,0,0])
c = Matrix([6,14,13])
r = Matrix([0,-1,0])

B = [0,1,2]

A_B = A[B,:]
b_B = b[B,:]

x =  A_B.solve(b_B)
l = A_B.transpose().solve(c)
d = A_B.transpose().solve(r)
```

$$A_B \cdot x^*_B = b_B$$

$$x^*_B = A_B^{-1} \cdot b_B$$

$\{1,2,3\} = B$

x^*_B

λ second component is negative.

d

Example

$\lambda_B^T \cdot A_B = c^T$ $\quad$ $(A_B \cdot d) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; b = \begin{pmatrix} 10 \\ 14 \\ 11 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } c = \begin{pmatrix} 6 \\ 14 \\ 13 \end{pmatrix}$$

starting basis

$$B = \{1, 2, 3\}.$$

$$A_B = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \; b_B = \begin{pmatrix} 10 \\ 14 \\ 11 \end{pmatrix} \text{ and } x_B^* = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}.$$

$$\lambda_B = \begin{pmatrix} \frac{36}{5} \\ -\frac{4}{5} \\ \frac{1}{5} \end{pmatrix}, \; d = \begin{pmatrix} -\frac{2}{5} \\ \frac{3}{5} \\ -\frac{2}{5} \end{pmatrix}.$$

$$A \cdot d = \begin{pmatrix} 0 \\ -1 \\ 0 \\ \frac{2}{5} \\ -\frac{3}{5} \\ \frac{2}{5} \end{pmatrix}, \; b - Ax^* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 0 \\ 3 \end{pmatrix}.$$

2 leaves basis and 6 enters.

$B' = \{1, 3, 6\}$

Non-degenerate LPs

Definition

The LP $\max\{c^T x : x \in \mathbb{R}^n, Ax \le b\}$ is non-degenerate if number of zero-components in $Ax - b \in \mathbb{R}^m$ is at most n for each $x \in \mathbb{R}^n$.

Termination non-degenerate case

Theorem

If the linear program is non-degenerate, then the simplex algorithm terminates.

Proof: ε^* is always strictly positive.

x^*_B

$x^*_B + \varepsilon^* \cdot d$

otherwise degenerate. (x^*_B sat all constraints in B and at least one other one with equality.)

$B \to B' \to \cdots$ strict improvement.

Never re-visit a basis. $\Rightarrow$ Alg. terminates.

Smallest index rule

several choices for i and j

$\lambda_i < 0$

j index where min $\frac{(Ax_B^*-b)_k}{(A\cdot d)_k}$ is attained.

Start with feasible basis B

`while` B is not optimal

Compute $\lambda \in \mathbb{R}^n$ such that $\lambda^T A_B = c^T$

Let $i^* \in B$ be the smallest index with $\lambda_i < 0$

Compute $d \in \mathbb{R}^n$ with $a_j^T d = 0, j \in B \setminus \{i^*\}$ and $a_{i^*}^T d = -1$

Determine $K = \{k \colon 1 \leq k \leq m, \, a_k^T d > 0\}$

`if` $K = \emptyset$

assert LP unbounded

`else`

Let $k^* \in K$ the smallest index where $\min\limits_{k \in K}(b_k - a_k^T x^*)/a_k^T d$ is attained

update $B := B \setminus \{i^*\} \cup \{k^*\}$

Termination

Theorem

The simplex algorithm with the smallest index rule terminates.

Proof: Suppose not. Then $B_0, \ldots, B_s$ bases visited, with $B_s = B_0$.

Let j be the largest index that ever enters or leaves. Since $B_0 = B_s$ each index that leaves at some point enters and vice versa.

Let p be index in which j leaves $\quad j \uparrow\downarrow B_p \quad \lambda^{(p)}$ λ at ~~A~~ B_p

— q ———— j enters. $\quad j \downarrow\uparrow B_q \quad d^{(q)}$ d at B_q

$$\underbrace{\lambda^{(p)T} \cdot A}_{c^T} \cdot d^{(q)} > 0$$

Let i be index in B_p s.th.

$$\lambda_i \, a_i^T \cdot d^{(q)} > 0$$

CASE 2: $i > j$

$i \in B_p$ and $i \in B_q$

But $a_i^T \cdot d^{(q)} = \boxed{0}$ -1 ~~as~~ cannot be since i would leave B^q

then. $\lambda_i a_i^T d^{(q)} = 0$ $\lightning$

Case 1: $i = j$ $\quad \lambda_i < 0 \quad a_i^T \cdot d^{(q)} > 0$ $\lightning$

if $i < j$ $\lambda_i^{(p)} > 0$ $a_i^T \cdot d^{(q)} > 0$ not possible!

Remember $\varepsilon^* = 0$ during the whole cycle. If $a_i^T d^{(q)} > 0$, then i would enter B_q.