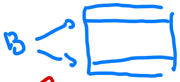
Review: The simplex algorithm

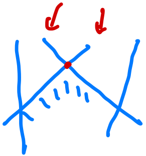
$$\text{MAX } c^T x \qquad Ax \le b$$

$A \in \mathbb{R}^{m \times n}$, $\operatorname{rank}(A) = n$

Given: $B \subseteq \{1, \dots, m\}$ feasible basis.



n lin. indep. rows



$$x_B^* = A_B^{-1} \cdot b_B$$

Recall: B is optimal basis if B is feasible and $\lambda_B \geq 0$, where

$$\lambda_B^T \cdot A_B = c^T$$

unique solution.

If B is optimal, then x_B^* is opt. sol. of LP.

$$\max\ c^T x \quad Ax \le b \qquad \le \qquad \max\ c^T x \quad A_B x \le b_B$$

x_B^* is feas

x_B^* feas

x_B^* is optimal: z any feas. sol

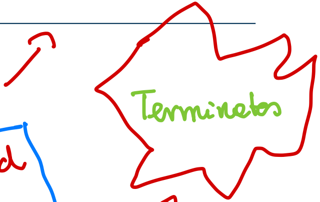
$$c^T z = \lambda_B^T \cdot \underbrace{A_B \cdot z}_{\le b_B} \le \lambda_B^T \cdot b_B = \underbrace{\lambda_B^T \cdot A_B}_{c^T} \cdot x_B^*$$

$$\Rightarrow c^T z \le c^T x_B^*$$

Review: The simplex algorithm

$\overset{i}{\uparrow}\ \overset{j}{\downarrow}$ B

B feasible basis.



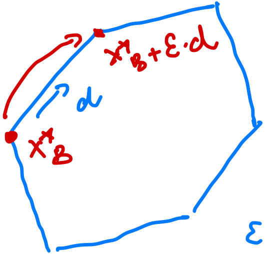
P

Last time.

Compute λ_B : $\lambda_B^T \cdot A_B = c^T$ [Gauss]

If $\lambda_B \geq 0 \Rightarrow B$ is optimal STOP

Let $i \in B$ smallest index with $\lambda_i < 0$



$\varepsilon \geq 0$

Compute: $d \in \mathbb{R}^n$ s.th. $A_B \cdot d = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$
[Gauss]

$K = \{ j \in \{1, \dots, m\} : (A \cdot d)_j > 0 \}$

$A(x^* + \varepsilon \cdot d) \leq b$

$\Leftrightarrow \forall j \in K: \quad \varepsilon \leq \dfrac{(b - Ax^*)_j}{(A \cdot d)_j}$ $j \in K$

Set $B := B \setminus \{i\} \cup \{j\}$.

let j smallest index, where min is attained

Finding an initial feasible basis

$\max c^T \cdot x$
$Ax \le b$
$x \in \mathbb{R}^n$

If x^* is feas. for (4) then $x^*, \lambda = 0$ feas. for $(\overline{LP})$ $\iff$ $OPT_{\overline{LP}} = 0$

$x = x^+ - x^-$, $x^+, x^- \in \mathbb{R}^n_{\geq 0}$

Linear program has equivalent form

$\approx$ $\max \ c^T x^+ - c^T \cdot x^-$

$$\max\{c^T x : Ax \le b, \ x \geqslant 0\}. \qquad (4)$$

$A \cdot x^+ - A \cdot x^- \le b$

$x^+, x^- \geq 0$

Task: Find initial feasible Basis for (4).

Split $Ax \le b$ as

$A_1 x \le b_1$ and $A_2 x \le b_2$

with $b_1 \geqslant 0$ and $b_2 < 0$.

$(\overline{LP})$

Consider: $\min \ \lambda_1 + \dots + \lambda_k$

$A_1 \cdot x \le b_1$

$A_2 \cdot x \le b_2 + \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix}$

Basis

$(\lambda \le |b_2|)$ $(x \geq 0)$, $\lambda \geq 0$

Set $x^* = 0$

$\lambda = |b_2|$

initial basic feas. sol.

$Ax \le b \ \begin{cases} A_1 x \le b_1 \\ A_2 x \le b_2 \end{cases}$

$A_2 \in \mathbb{R}^{k \times n}$

Finding an initial feasible basis

Linear program has equivalent form

$$\max\{c^T x \colon Ax \leq b,\, x \geqslant 0\}. \qquad (4)$$

Split $Ax \leq b$ as

$A_1 x \leq b_1$ and $A_2 x \leq b_2$

with $b_1 \geqslant 0$ and $b_2 < 0$.

Exercise: Let $B \subseteq \{1, \dots, m+k\}$ be opt. basis of $\overline{LP}$ with OPT VALUE 0 $\Rightarrow$ $B \cap \{1, \dots, m\}$ is initial feas basis for (4)

$\overline{LP}$

$$\min \lambda_1 + \dots + \lambda_k$$

$$A_1 x \leq b_1$$

$$A_2 x - I_k \lambda \leq b_2$$

$$-I x \leq 0$$

$$-I\lambda \leq 0, \quad I\lambda \leq |b_2|$$

B

$B \cap \{1, \dots, m\}$

$$\begin{array}{cc} \overbrace{\qquad}^{n \text{ col.}} & \\ A_1 & \\ -I_n & \\ A_2 & -I_k \\ & -I_k \end{array}$$

$n+k$ lin. ind. rows.

Duality

$A \cdot x^* \leq b$

OPT

corresponding upper bound

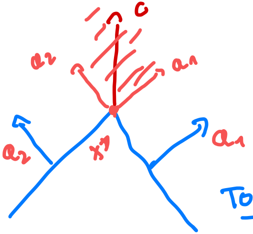
C is conic combination of rows(A)

$$\max\{c^T x : x \in \mathbb{R}^n,\ Ax \leq b\},$$

(Primal)

$$\min\{b^T y : y \in \mathbb{R}^m,\ A^T y = c,\ y \geqslant 0\}.$$

(Dual)



Weak DUALITY: max $\leq$ min.

Let x^*, y^* be feas for (P) and (D) respectively

To show: $c^T \cdot x^* \leq b^T y^*$

$\leq b^T$

$x^{*T} \cdot c = x^{*T} \cdot A^T \cdot y^* \leq b^T \cdot y^*$ $\blacksquare$

Duality

Weak Duality

Theorem (Weak duality)

If x^* and y^* are primal and dual feasible solutions respectively, then $c^T x^* \leq b^T y^*$.

In other words $(P) \leq (D)$

Strong duality

Theorem

If the primal linear program is feasible and bounded, then so is the dual linear program. Furthermore in this case, both linear programs have an optimal solution and the optimal values coincide.

(P)

$$\max \; c^T \cdot x$$
$$Ax \leq b$$

(D)

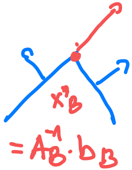
$$\min \; b^T \cdot y$$
$$A^T \cdot y = c$$
$$y \geq 0$$

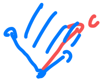
$A \in \mathbb{R}^{m \times n}$.

Proof: CASE 1 $\quad \operatorname{rank}(A) = n$

USE SIMPLEX To find optimal BASIS $B \subseteq \{1, \dots, m\}$.

Since B is optimal: $\quad \lambda_B \geq 0$





One has:

$$
\begin{aligned}
c^T \cdot x^*_B &= \lambda_B^T \cdot A_B \cdot x^*_B \\
&= \lambda_B^T A_B A_B^{-1} \cdot b_B \\
&= \lambda_B^T \cdot b_B \\
&= \lambda^T \cdot b
\end{aligned}
$$

Feas for (D) with $\lambda_i = \begin{cases} 0 & i \notin B \\ (\lambda_B)_i & i \in B. \end{cases}$

(P)

max $c^T \cdot x$

$Ax \le b$

(D)

min $b^T \cdot y$

$A^T \cdot y = c$

$y \ge 0$

$A \in \mathbb{R}^{m \times n}$.

<u>CASE 2:</u> rank $(A) < n$

<u>Re-write:</u> (P)

$(\tilde{P})$
$$\begin{cases} \max \; c^T x^+ - c^T \cdot x^- \\ A x^+ - A x^- \le b \; \} y_1 \\ -I x^+ \le 0 \; \} y_2 \\ -I x^- \le 0 \; \} y_3 \end{cases}$$

$$\left[\begin{array}{c|c} A & -A \\ \hline -I & 0 \\ \hline 0 & -I \end{array}\right]$$

has rank $2n$ = # VAR.

Transp.

$$\begin{array}{ccc} y_1 & y_2 & y_3 \\ \left[\begin{matrix} A^T & -I & 0 \\ -A^T & 0 & -I \end{matrix}\right] \end{array}$$

Construct DUAL OF $(\tilde{P})$

$(\tilde{D})$:

DROP

min: $b^T \cdot y_1 + \boxed{0^T \cdot y_2 + 0^T \cdot y_3}$

$A^T \cdot y_1 - y_2 \overset{\ge}{=} c^T$

$-A^T \cdot y_1 - y_3 \overset{\ge}{=} -c^T$

$y_1, y_2, y_3 \ge 0$

together. $A^T \cdot y_1 = c^T$

$\Rightarrow (\tilde{D}) \sim (D)$

CASE 1 settles also CASE 2. ☒

Dual of the Dual

(P) $\max c^T x$
$Ax \leq b$

TRANSFORM (D) into $\leq$ Standard form

interpret $\underline{x = \lambda_1 - \lambda_2}$

(D)

$$\min\{b^T y : y \in \mathbb{R}^m, A^T y = c, y \geqslant 0\}$$

$c^T x$

DUAL

$= - \max -b^T \cdot y$

$A^T \cdot y \leq c$

$-A^T \cdot y \leq -c$

$-I y \leq 0$

$\sim (-) \min \overbrace{c^T \cdot \lambda_1 - c^T \cdot \lambda_2} + 0^T \cdot \lambda_3$

$\underbrace{A \cdot \lambda_1 - A \cdot \lambda_2 - I \lambda_3 = -b}$

$\lambda_1, \lambda_2, \lambda_3 \geq 0$

$A \cdot x \geq -b$

$$-\min c^T \cdot x$$
$$A \cdot x \geq -b \quad |\cdot(-1)$$

$\sim$

$$-\min c^T \cdot x$$
$$A(-x) \leq b$$

$\sim$

$$\max c^T(-x)$$
$$A(-x) \leq b$$

Substitute $-x \to x$

$\sim$

$$\max c^T \cdot x$$
$$Ax \leq b.$$

$$-\max f(x)$$
$$= \min -f(x)$$

$$-\min c^T \cdot x$$
$$= \max c^T(-x)$$

Dual of Dual $\approx$ Primal. $\blacksquare$

10 Minutes break for Retour Indicatif

Dual of the Dual

Corollary

If the dual linear program has an optimal solution, then so does the primal linear program and the objective values coincide.

Table of possibilities

Further example

$$\begin{array}{rcl} \max & c^T x & \\ Bx & = & b \\ Cx & \leq & d. \end{array}$$

(Primal)

$$\begin{array}{rcl} & \min b^T y_1 + d^T y_2 & \\ B^T y_1 + C^T y_2 & = & c \\ y_2 & \geqslant & 0. \end{array}$$

(Dual)

Zero sum games

$$A = \begin{pmatrix} 5 & 1 & 3 \\ 3 & 2 & 4 \\ -3 & 0 & 1 \end{pmatrix}$$

Row player: Chooses row i

Column player: Chooses column j

Rock – paper – scissors

Deterministic strategies

$$\max_i \min_j$$

$$\max_j \min_i$$

Mixed strategies

Definition

Let $A \in \mathbb{R}^{m \times n}$ define a two-player matrix game. A mixed strategy for the row-player is a vector $x \in \mathbb{R}^m_{\geqslant 0}$ with $\sum_{i=1}^m x_i = 1$. A mixed strategy for the column player is a vector $y \in \mathbb{R}^n_{\geqslant 0}$ with $\sum_{j=1}^n y_i = 1$.

$$E[\text{Payoff}] = x^T A y. \tag{5}$$

Example: Mixed strategy rock – paper – scissors

Weak duality

Lemma

Let $A \in \mathbb{R}^{m \times n}$, then

$$\max_{x \in X} \min_{y \in Y} x^T A y \leq \min_{y \in Y} \max_{x \in X} x^T A y,$$

where X and Y denote the set of mixed row and column-strategies respectively.

Minimax-Theorem

Theorem (von Neumann (1928))

$$\max_{x \in X} \min_{y \in Y} x^T A y = \min_{y \in Y} \max_{x \in X} x^T A y,$$

where X and Y denote the set of mixed row and column-strategies respectively.