Linear, affine, conic and convex hulls

Let $X \subseteq \mathbb{R}^n$:

$$
\begin{aligned}
\operatorname{lin.hull}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \in \mathbb{N}_0, \\
&\qquad x_1, \ldots, x_t \in X,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}\} \\
\operatorname{affine.hull}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \in \mathbb{N}_+, \\
&\qquad x_1, \ldots, x_t \in X,\ \sum_{i=1}^{t} \lambda_i = 1,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}\} \\
\operatorname{cone}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \in \mathbb{N}_0, \\
&\qquad x_1, \ldots, x_t \in X,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}_{\geqslant 0}\} \\
\operatorname{conv}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \in \mathbb{N}_+, \\
&\qquad x_1, \ldots, x_t \in X,\ \sum_{i=1}^{t} \lambda_i = 1,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}_{\geqslant 0}\}
\end{aligned}
$$

Farkas' Lemma:

$Ax \leq b$

not feas. $\Rightarrow$ $\lambda \in \mathbb{R}^m_{\geq 0}$

s. th. $\lambda^T A = 0$

$\lambda^T \cdot b = -1$

Linear, affine, conic and convex hulls

Let $X \subseteq \mathbb{R}^n$:

$$
\begin{aligned}
\mathrm{lin.\,hull}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \in \mathbb{N}_0,\\
&\qquad x_1, \ldots, x_t \in X,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}\}\\
\mathrm{affine.\,hull}(X) &= \{\lambda_1 x_1 + \cdots + \lambda_t x_t \mid t \in \mathbb{N}_+,\\
&\qquad x_1, \ldots, x_t \in X,\ \sum_{i=1}^{t} \lambda_i = 1,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R}\}\\
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\end{aligned}
$$

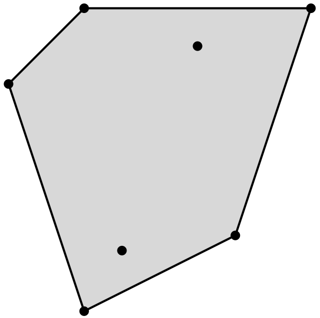


Figure: The convex hull of 7 points in $\mathbb{R}^2$.

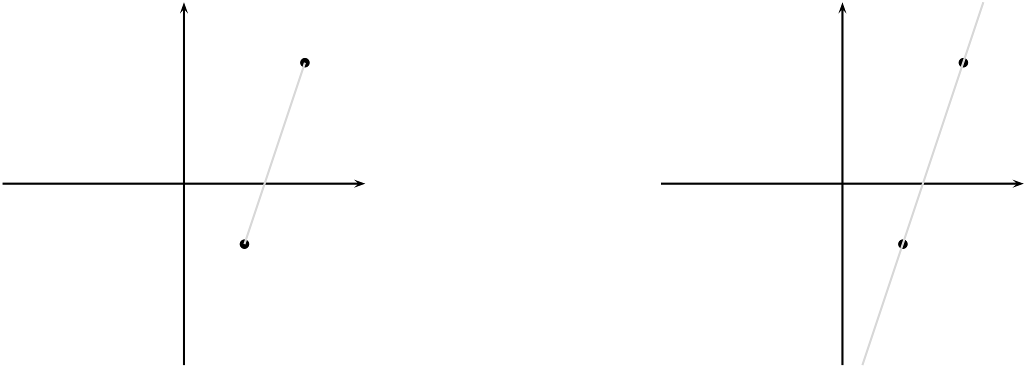


Figure: Two points with their convex hull on the left and their affine hull on the right.

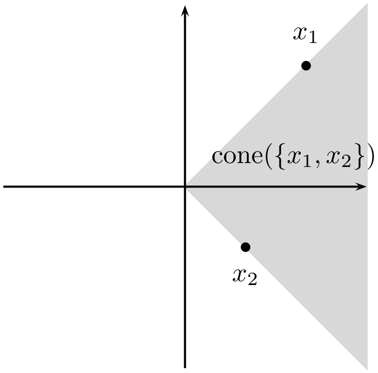


Figure: Two points with their conic hull

Linear and affine hulls

$X - x_0 = \{x - x_0 : x \in X\}$

Theorem

Let $X \subseteq \mathbb{R}^n$ and $x_0 \in X$. One has

$$\text{affine. hull}(X) = x_0 + \text{lin. hull}(X - x_0),$$

where for $u \in \mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$, $u + V$ denotes the set $u + V = \{u + v \mid v \in V\}$.

"$\subseteq$" Let $y \in$ affine. hull (X). This means $\exists\, t \in \mathbb{N}_+$ $x_1, \dots, x_t \in X$ and $\lambda_1, \dots, \lambda_t \in \mathbb{R}$ with $\sum \lambda_i = 1$ and $y = \sum_{i=1}^{t} \lambda_i x_i$

$$y = \sum_{i=1}^{t} \lambda_i x_i - x_0 + x_0 \qquad \sum \lambda_i = 1$$

$$= \underbrace{\sum_{i=1}^{t} \lambda_i (x_i - x_0)}_{\in \text{ lin. hull}(X - x_0)} + x_0$$

"$\supseteq$" Let $y \in x_0 + \text{lin. hull}(X - x_0)$. This means there exists $t \in \mathbb{N}_0$, $x_1, \dots, x_t \in X$ s.th. $y = x_0 + \sum_{i=1}^{t} \lambda_i (x_i - x_0)$

and $\lambda_1, \dots, \lambda_t \in \mathbb{R}$

$$= \sum_{i=1}^{t} \lambda_i x_i + x_0 - \sum_{i=1}^{t} \lambda_i \cdot x_0$$

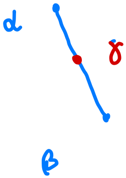
$$= \sum_{i=1}^{t} \lambda_i x_i + \left(1 - \sum_{i=1}^{t} \lambda_i\right) x_0 \quad \text{showing } \in \text{affine. hull}(X) \quad \blacksquare$$

Convex hull is convex

$$\mathrm{conv}(X) = \left\{ \lambda_1 x_1 + \dots + \lambda_t x_t : t \in \mathbb{N}_+ ,\ \lambda_1, \dots, \lambda_t \geq 0,\ \sum_{i=1}^{t} \lambda_i = 1 \right\}$$

Theorem

Let $X \subseteq \mathbb{R}^n$ be a set of points. The convex hull, $\mathrm{conv}(X)$, of X is convex.



$\gamma \cdot \alpha + (1-\gamma) \cdot \beta$ for $\gamma \in [0,1]$

<u>To show:</u> if $\alpha, \beta \in \mathrm{conv}(X)$, then $\bullet \in \mathrm{conv}(X)$

Proof: $\alpha = \sum_{i=1}^{t} \lambda_i \cdot x_i \qquad \beta = \sum_{i=1}^{t} \mu_i x_i$ with $\mu, \lambda \in \mathbb{R}^t_{\geq 0}$

and $\sum \lambda_i = \sum \mu_i = 1$

Let $\gamma \in [0,1]$ and consider

$$\gamma \cdot \alpha + (1-\gamma) \cdot \beta$$

$$= \sum_{i=1}^{t} \left(\gamma \cdot \lambda_i + (1-\gamma) \cdot \mu_i\right) x_i$$

But $\sum_{i=1}^{t} \underbrace{\gamma \cdot \lambda_i}_{\geq 0} + \underbrace{(1-\gamma)\mu_i}_{\geq 0}$

$= \gamma \cdot \sum_{i=1}^{t} \lambda_i + (1-\gamma) \sum_{i=1}^{t} \mu_i$

$= 1$

$\alpha \in \mathrm{conv}(X_1)$ — finite set $\subseteq X$

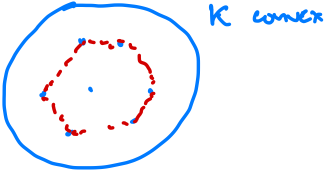
$\beta \in \mathrm{conv}(X_2)$ — finite set $\subseteq X$

$\alpha, \beta \in \mathrm{conv}(X_1 \cup X_2)$

Convex hull is minimal

Theorem

Let $X \subseteq \mathbb{R}^n$ be a set of points. Each convex set K containing X also contains $\mathrm{conv}(X)$.



proof: to show for $x_1, \dots, x_t \in X$ and $\lambda_1, \dots, \lambda_t \geq 0$ with $\sum \lambda_i = 1$

$$\Rightarrow \sum_{i=1}^{t} \lambda_i x_i \in K$$

Induction: $t=1$ follows from $X \subseteq K$

$t>1$
$$\sum_{i=1}^{t} \lambda_i x_i = \lambda_1 \cdot x_1 + \sum_{i=2}^{t} \lambda_i x_i = x$$

$$= \frac{1}{(1-\lambda_1)} \cdot \underbrace{\sum_{i=2}^{t} \lambda_i}_{= (1-\lambda_1)} = 1$$

if $\lambda_1 = 0$, then by induction $x \in K$
or $\lambda_1 = 1$

otherwise: $x = \lambda_1 \cdot x_1 + (1-\lambda_1) \boxed{\sum_{i=2}^{t} \frac{\lambda_i}{\underbrace{(1-\lambda_1)}_{\geq 0}} \cdot x_i} \in K$ by induction.

then $x \in K$ by def. of convexity ■

Corollary

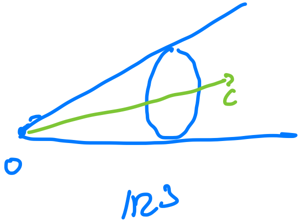
$$\operatorname{conv}(X) = \bigcap_{\substack{K \supseteq X \\ K \text{ convex}}} K .$$

Cones

OBSERVATION: if $C \subseteq \mathbb{R}^n$ is cone, $C \neq \emptyset$, then $0 \in C$.

Definition

A set $C \subseteq \mathbb{R}^n$ is a cone, if it is convex and for each $c \in C$ and each $\lambda \in \mathbb{R}_{\geqslant 0}$ one has $\lambda \cdot c \in C$.



Analogous theorems for cones

$y \in \operatorname{cone}(X)$

$\beta \cdot y = \sum_{i=1}^{t} \underbrace{(\beta \cdot \lambda_i)}_{\in \mathbb{R}_{\geq 0}} \cdot x_i$ for $\lambda \in \mathbb{R}^t_{\geq 0}$, $x_1, \ldots, x_t \in X$

$\beta \in \mathbb{R}_{\geq 0}$

Theorem

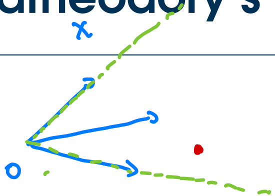
For any $X \subseteq \mathbb{R}^n$, the set $\operatorname{cone}(X)$ is a cone.

Theorem

Let $X \subseteq \mathbb{R}^n$ be a set of points. Each cone containing X also contains $\operatorname{cone}(X)$.

$$\operatorname{cone}(X) = \bigcap_{\substack{C \supseteq X \\ C \text{ is a cone}}} C.$$

Carathéodory's Theorem



Theorem

Let $X \subseteq \mathbb{R}^n$, then for each $x \in \text{cone}(X)$ there exists a set $\widetilde{X} \subseteq X$ of cardinality at most n such that $x \in \text{cone}(\widetilde{X})$. The vectors in $\widetilde{X}$ are linearly independent.

<u>Proof:</u> Let $x \in \text{cone}(X)$. This means that there exists $t \in \mathbb{N}_0$, $\underbrace{x_1, \dots, x_t}_{:= \tilde{X}} \in X$

$\lambda_1, \dots, \lambda_t \in \mathbb{R}_{\geq 0}$ with $x = \sum_{i=1}^{t} \lambda_i \cdot x_i$. If $\lambda_i = 0$ for some $i \in \{1, \dots, t\}$ then $x \in \text{cone}(\tilde{X} - x_i)$

We assume therefore $\lambda_1, \dots, \lambda_t > 0$. We now show: if $\tilde{X}$ lin. dependent,

$\Rightarrow \exists\, i \in \{1, \dots, t\}$ s.th. $x = \text{cone}(\tilde{X} - x_i)$

if $\vec{x}$ lin dependent, then $\exists$ $\beta_1, \dots, \beta_t \in \mathbb{R}$ not all zero s.th.

$$\beta_1 x_1 + \dots + \beta_t x_t = 0 \quad , \quad \Rightarrow \forall \varepsilon > 0 : \ \varepsilon \cdot \beta_1 x_1 + \dots + \varepsilon \cdot \beta_t \cdot x_t = 0$$

W.L.o.G: $\exists i : \beta_i < 0$

$$x = (\lambda_1 + \varepsilon \beta_1) x_1 + \underbrace{(\lambda_2 + \varepsilon \beta_2)}_{\text{Goal } \geq 0} x_2 + \dots (\lambda_i + \varepsilon \cdot \underbrace{\beta_i}_{<0}) x_i \ \dots + (\lambda_t + \varepsilon \beta_t) x_t$$

$\forall i$: $\quad \lambda_i + \varepsilon \cdot \beta_i \geq 0 \iff \lambda_i \geq -\varepsilon \cdot \beta_i \quad \forall i \in \{1, \dots, t\}$

$\iff \lambda_i \geq -\varepsilon \cdot \beta_i \quad i \in \{1, \dots, t : \beta_i < 0\}$

$\iff \varepsilon \leq \dfrac{\lambda_i}{-\beta_i} \quad i \in \underbrace{\{1, \dots, t : \beta_i < 0\}}_{I^-}$.

__Let:__ $j \in \{1, \dots, t : \beta_i < 0\}$ be index where $\min\limits_{i \in I^-} \left\{ \dfrac{\lambda_i}{-\beta_i} \right\}$ is attained.

let ε^* be the corresponding value : $\dfrac{\lambda_j}{-\beta_j} = \varepsilon^*$.

$\Rightarrow$ With ε^*:

$$X = \underbrace{(\lambda_1 + \varepsilon^* \cdot \beta_1)}_{\geq 0} x_1 + \dots + \underbrace{(\lambda_j + \varepsilon^* \cdot \beta_j)}_{= 0} x_j + \dots + \underbrace{(\lambda_t + \varepsilon^* \beta_t)}_{\geq 0} x_t$$

is conic combination of the set $\{x_1, \dots, x_t\} \setminus \{x_j\}$.

$\boxtimes$

Bounded continuous functions

Theorem

Let $X \subseteq \mathbb{R}^n$ be compact and $f : X \to \mathbb{R}$ be continuous. Then f is bounded and there exist points $x_1, x_2 \in X$ with $f(x_1) = \sup\{f(x) \colon x \in X\}$ and $f(x_2) = \inf\{f(x) \colon x \in X\}$.

Separation theorem

Theorem

Let $K \subseteq \mathbb{R}^n$ be a closed convex set and $x^* \in \mathbb{R}^n \setminus K$, then there exists an inequality $a^T x \leq \beta$ such that $a^T y < \beta$ holds for all $y \in K$ and $a^T x^* > \beta$.

$a^T \cdot x = \beta$

$a^T \cdot x^* > \beta$

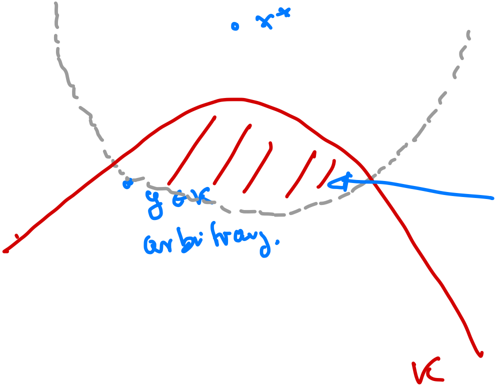
$a^T \cdot y < \beta \quad \forall y \in K$

<u>Proof:</u> $K \subseteq \mathbb{R}^n$ is convex and closed. $x^* \in K$.

$\min_{x \in K} \|x^* - x\|$ is attained.

$f: \mathbb{R}^n \to \mathbb{R}$

$x \longmapsto \|x^* - x\|$ is continuous.

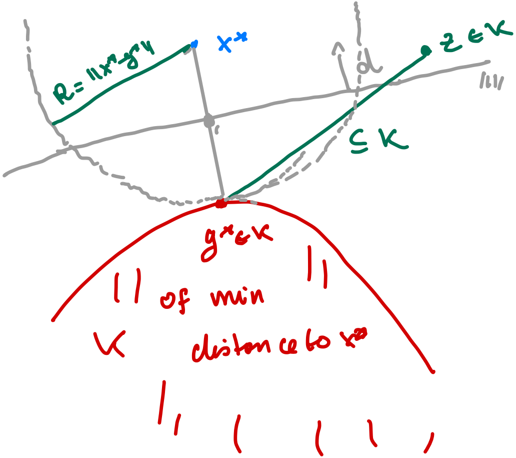


$B \subseteq \mathbb{R}^n$. $B = \{x \in \mathbb{R}^n : \|x - x^*\| \leq \|y - x^*\|\}$.

$K \cap B$

is convex ~~&~~ closed and bounded $\Rightarrow$ compact.

$\Rightarrow$ min is attained.



$d^T \cdot x = \beta$

$$d = \frac{(x^* - y^*)}{\|x^* - y^*\|} \qquad \beta = d^T \frac{(x^* - y^*)}{2}$$

Suppose $d^T \cdot x = \beta$ is not separating.

then: $\exists\, z \in K$ s.th. $d^T \cdot z \geq \beta$

↯ to $y^* \in K$ of min. distance to x^*.

Exercise: Formal proof of ↯.

Farkas' Lemma – Version 1

Theorem (Farkas' lemma)

Let $A \in \mathbb{R}^{m \times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax = b$, $x \geqslant 0$ has a solution if and only if for all $\lambda \in \mathbb{R}^m$ with $\lambda^T A \geqslant 0$ one has $\lambda^T b \geqslant 0$.

Proof: „$\Rightarrow$" Let $x^* \in \mathbb{R}^n_{\geq 0}$ be a solution. Then with $\lambda \in \mathbb{R}^m$ and $\lambda^T \cdot A \geq 0$, one has

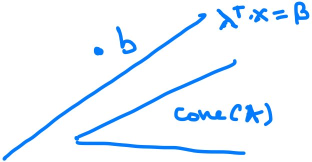
$$\lambda^T \cdot b = \lambda^T \cdot (A \cdot x^*) = \underbrace{\underbrace{(\lambda^T \cdot A)}_{\geq 0} \underbrace{x^*}_{\geq 0}}_{\geq 0}$$

"$\Leftarrow$" if $Ax=b$, $x \geq 0$ has no solution. We show: $\exists\, \lambda \in \mathbb{R}^m$

with: $\lambda^T \cdot A \geq 0$ and $\lambda^T \cdot b < 0$

$\text{cone}(A) = \{ A \cdot x : x \in \mathbb{R}^n_{\geq 0} \}$. cone generated by columns of A.

Exercise: cone(A) is closed. (and convex).



$\lambda^T \cdot b > \beta \geq 0$ since $0 \in \text{cone}(A)$

$\forall$ column $a_i \in A$ ($\in \text{cone}(A)$)

we have $\lambda^T \cdot a_i \leq 0$

Since if $\lambda^T \cdot a_i > 0$, then for $\gamma > 0$ large enough. $\lambda^T(\underbrace{\gamma \cdot a_i}_{\in \text{cone}}) > \beta$ ↯.

$\lambda^T \cdot b > 0$

and $\lambda^T \cdot A \leq 0$

∎

Farkas' Lemma – Version 2

Theorem (Farkas' lemma)

Let $A \in \mathbb{R}^{m \times n}$ be a matrix and $b \in \mathbb{R}^m$ be a vector. The system $Ax \leq b$ has a solution if and only if for all $\lambda \in \mathbb{R}^m_{\geqslant 0}$ with $\lambda^T A = 0$ one has $\lambda^T b \geqslant 0$.

Exercise!

$Ax \leq b$ sol.

$$\left.\begin{array}{l} A(x^+ - x^-) \leq b \\ x^+, x^- \geq 0 \end{array}\right\} \text{sol.}$$

$A(x^+ - x^-) + s = b$

$x^+, x^-, s \geq 0$

has solution.