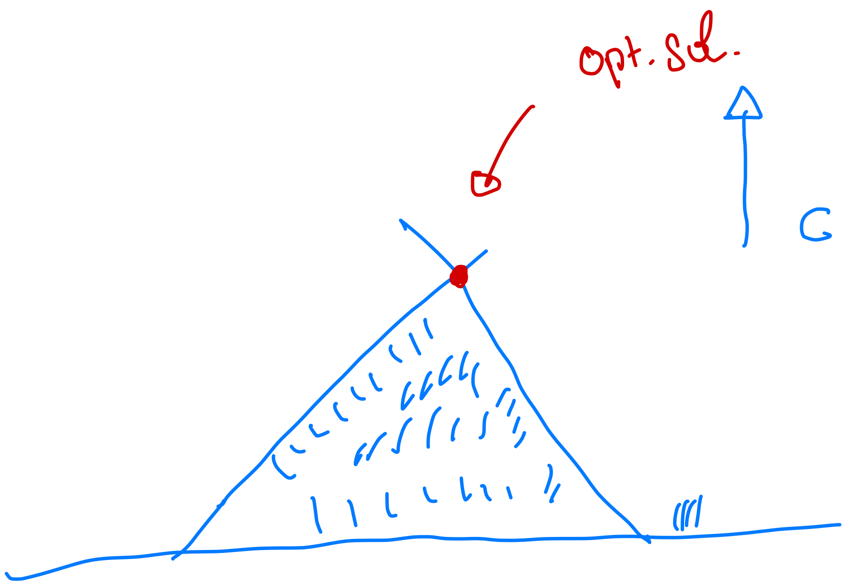
Recap:

Linear program

$$\max \; c^T \cdot x$$

subject to: $Ax \leq b$ $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$



Today:

- If LP has corners and is bounded $\Rightarrow$ ~~co~~ $\exists$ opt. sol that is a "corner"
- # corners is finite
- enumeration

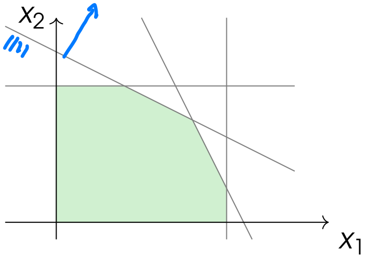
$\Rightarrow$ First **very slow** algorithm to solve LP.

Polyhedra

Definition

A polyhedron $P \subseteq \mathbb{R}^n$ is a set of the form $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ for some $A \in \mathbb{R}^{m \times n}$ and some $b \in \mathbb{R}^m$.

$$A = \begin{pmatrix} 3 & 6 \\ 8 & 4 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 30 \\ 44 \\ 5 \\ 4 \\ 0 \\ 0 \end{pmatrix}:$$



$Ax \leq b$

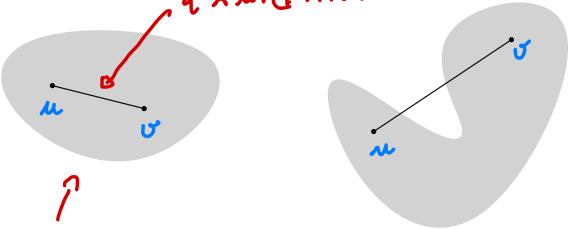
Set of feasible solutions of previous example
"Soft DRINK PRODUCTION"

Convex sets

Definition

A set $K \subseteq \mathbb{R}^n$ is convex if for each $u, v \in K$ and $\lambda \in [0, 1]$ the point $\lambda u + (1 - \lambda)v$ is also contained in K.

$\{ \lambda \cdot u + (1-\lambda) \cdot v \ : \ \lambda \in [0,1] \}$



Convex

NOT convex

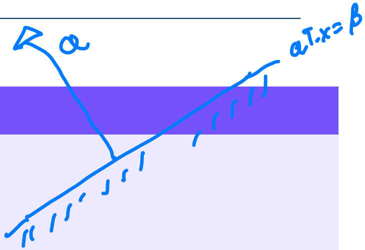
Halfspaces

Definition

A halfspace is a set of the form

given: $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}$.

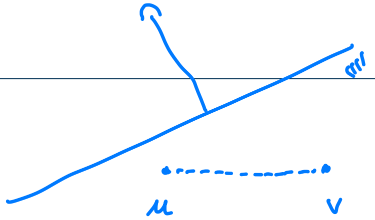
$$\{x \in \mathbb{R}^n : a^T x \leq \beta\}.$$



A hyperplane is a set of the form

$$\{x \in \mathbb{R}^n : a^T x = \beta\}.$$

Halfspaces are convex



Lemma

A half-space is convex.

proof: Let $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ define $\{x \in \mathbb{R}^n : a^T \cdot x \le \beta\}$

Let $u \in \mathbb{R}^n$, $v \in \mathbb{R}^n$ s.th. $a^T \cdot u \le \beta$, $a^T \cdot v \le \beta$ (u, v in half-space)

$\lambda \in [0,1]$. $a^T(\lambda \cdot u + (1-\lambda) \cdot v)$

$= \lambda \cdot \overbrace{a^T \cdot u}^{\le \beta} + (1-\lambda) \overbrace{a^T \cdot v}^{\le \beta} \le \lambda \cdot \beta + (1-\lambda) \cdot \beta = \beta \Rightarrow$ $\lambda \cdot u + (1-\lambda) v$ in halfspace.

Intersections of convex sets

Lemma

Let I be an index set and $C_i \subseteq \mathbb{R}^n$ be convex sets for each $i \in I$, then $\cap_{i \in I} C_i$ is a convex set. Exercise !

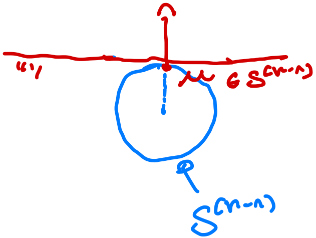
Corollary

A polyhedron is a convex set.

Polyhedron is intersection of half-spaces. $\Rightarrow$ convex.

Exercise: $B = \{ x \in \mathbb{R}^n : \|x\|_2 \leq 1 \}.$

B is intersection of (infinite) ~~...~~ set of half-spaces.



$\{ x \in \mathbb{R}^n : u^T \cdot x \leq 1 \}$

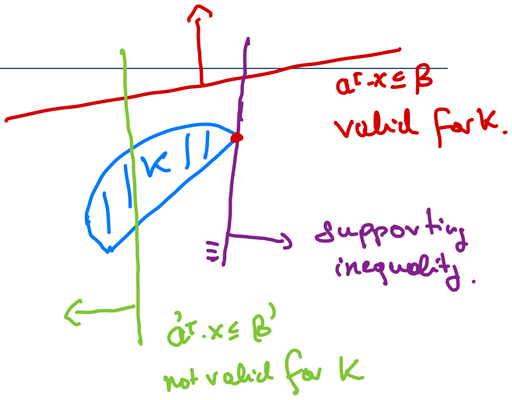
Valid inequalities

Definition

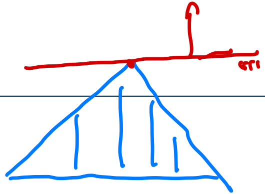
$a^T x \leq \beta$ is valid for $K \subseteq \mathbb{R}^n$ if for each $x^* \in K$:

$$a^T x^* \leq \beta$$

If in addition $(a^T x = \beta) \cap K \neq \emptyset$, then $a^T x \leq \beta$ is a supporting inequality and $a^T x = \beta$ is a supporting hyperplane



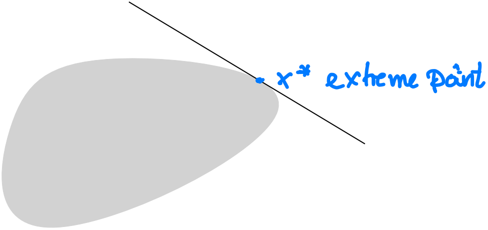
Extreme points



Definition

Let $K \subseteq \mathbb{R}^n$ be convex. $x^* \in K$ is extreme point or vertex of K if there exists a valid inequality $a^T x \leq \beta$ of K such that

$$\{x^*\} = K \cap \{x \in \mathbb{R}^n \colon a^T x = \beta\}.$$



Vertices of polyhedra – algebraic characterization

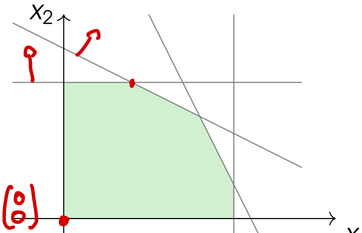
Theorem

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron. $x^* \in P$ is extreme point iff there is sub-system $A'x \leq b'$ of $Ax \leq b$ s.t.

i) *x^* satisfies all inequalities of $A'x \leq b'$ with equality.*

ii) *A' has n rows and A' is non-singular.*

$$A = \begin{pmatrix} 3 & 6 \\ 8 & 4 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 30 \\ 44 \\ 5 \\ 4 \\ 0 \\ 0 \end{pmatrix}:$$



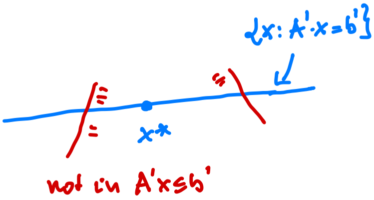
proof: "$\Rightarrow$" Let x^* satisfy $Ax \le b$ and

x^* extreme point of $P = \{x : Ax \le b\}$.

let $A'x \le b'$ be sub-system of $Ax \le b$ with $A'x^* = b'$

to show: $\operatorname{rank}(A') = n$. Let us suppose that $\operatorname{rank}(A') < n$.

Then $\exists\, d \in \mathbb{R}^n \setminus \{0\}$ s.th. $A' \cdot d = 0$

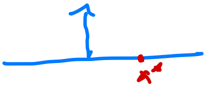


$\forall \varepsilon > 0$: $A'(x^* \pm \varepsilon \cdot d) = b'$

$\Rightarrow \exists \varepsilon > 0$ s.th. $(x^* \pm \varepsilon \cdot d) \in P$

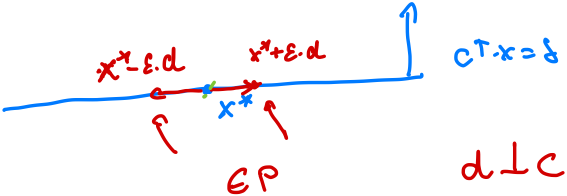
It remains to be shown: x^* is not an extreme point.

To this end: let



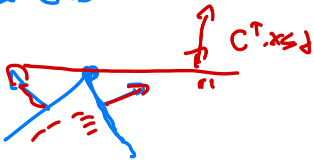
$c^T \cdot x \le \delta$ be a valid inequality for P

<u>to show:</u> $(c^T \cdot x = \delta) \cap P \neq \{x^*\}$



$d \perp c$ since $c^T \cdot x \leq \delta$ is valid for P and $x^* \pm \varepsilon \cdot d \in P$

"$\Leftarrow$" Let $x^* \in P$ and $A' x \leq b'$ sub-system of $Ax \leq b$ satisfied by x^* with equality and suppose A' ~~rows~~ $\in \mathbb{Q}^{k \times n}$ $\operatorname{rank}(A') = n$ to show: x^* is extreme point



"$\Leftarrow$" $A'x \le b'$ $\prec$ $Ax \le b$

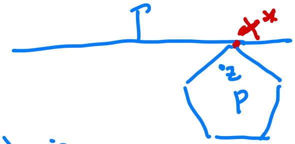
$\uparrow$ sub-system.

i) $A' \cdot x^* = b'$

ii) $A' \in \mathbb{R}^{n \times n}$,

iii) $\operatorname{rank}(A') = n$.

construct $c^T \cdot x \le \delta$



$c^T = \mathbb{1}^T \cdot A'$

$\delta = \mathbb{1}^T \cdot b'$

$\rightarrow$ $c^T x \le \delta$ is valid for P

<u>also:</u> $c^T \cdot x^* = \delta$

consider now: $z \neq x^* \in P$. to show: $c^T \cdot z < \delta$.

$A' \cdot z \le b'$, $\ne$ $\Rightarrow \exists\, y \in \mathbb{R}^n_{\ge 0}, y \neq 0$ s.th. $A' \cdot z = b' - y$

<u>Now:</u>

$$c^T \cdot z = \mathbb{1}^T \cdot A' \cdot z = \mathbb{1}^T (b' - y) = \underbrace{\mathbb{1}^T \cdot b'}_{= \delta} - \underbrace{\mathbb{1}^T \cdot y}_{> 0} < \delta \quad \blacksquare$$

Optimal solutions and vertices

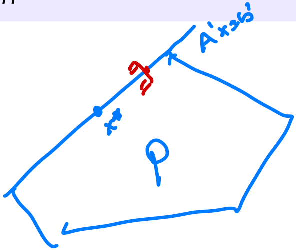
$$n\left\{\begin{matrix}\to\\\to\\\to\end{matrix}\right. \boxed{A}\, x \le \boxed{b}$$

$A'x = b'$

<u>Bounded:</u> $\exists\, M \in \mathbb{R}$ s.t. $c^T \cdot x \le M \quad \forall x \in P$

Theorem

If a linear program $\max\{c^T x : x \in \mathbb{R}^n, Ax \le b\}$ *is feasible and bounded and if* $\mathrm{rank}(A) = n$*, then the linear program has an optimal solution that is an extreme point.*



$P = \{x \in \mathbb{R}^n : Ax \le b\}.$

Proof: Let $\dim(x^*) := \operatorname{rank}(A')$ where $A'x \le b'$ is sub-system of $Ax \le b$ with $A'x^* = b'$ for $x^* \in P$. We now show: if $\dim(x^*) < n$ (i.e. x^* is not a vertex), then $\exists$ feasible $z \in P$ s.th.

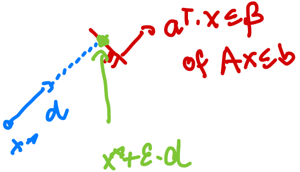
i) $\dim(z) > \dim(x^*)$

ii) $c^T z \ge c^T \cdot x^*$

Suppose $\dim(x^*) < n$.

then $\exists\, d \in \mathbb{R}^n \setminus \{0\}$ with $A'd = 0$

Case 1: $c^T \cdot d \ne 0$ W.l.o.G. $c^T \cdot d > 0$ (otherwise $d := -d$)



Let $\varepsilon > 0$ be maximal s.th. $x^* + \varepsilon \cdot d \in P$. Let $a^T \cdot x \le \beta$ be inequality of $(Ax \le b) \setminus (A'x \le b')$ with $a^T(x^* + \varepsilon \cdot d) = \beta$

then: a lin indep of rows of A' (otherwise $a^T \cdot d = 0$ ↯)

$\dim(x^* + \varepsilon \cdot d) > \dim(x^*)$ $\quad$ $x^* + \varepsilon \cdot d \in P$ and

$c^T(x^* + \varepsilon \cdot d) \geq c^T x^*$.

CASE 2: $\quad c^T \cdot d = 0$

$\in (Ax \leq b) \setminus (A'x \leq b')$

$A \cdot d \neq 0$ because $\operatorname{rank}(A) = n$

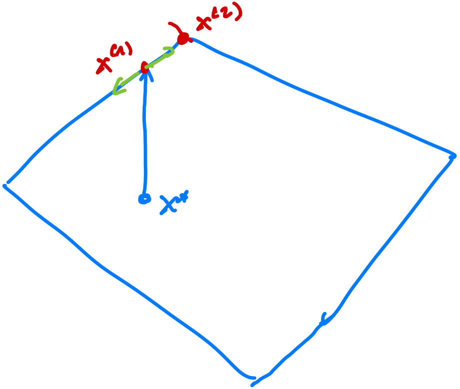
$\Rightarrow \exists$ inequality $a^T \cdot x \leq \beta$ of $Ax \leq b$ with $a^T \cdot d \neq 0$

w.l.g. $a^T \cdot d > 0$. Let ε^* be max s.th. $x^* + \varepsilon^* \cdot d \in P$.

Similar argument to CASE 1

$\Rightarrow$ $\dim(x^* + \varepsilon^* \cdot d) > \dim(x^*)$ and $c^T(x^* + \varepsilon \cdot d) = c^T x^*$ $\blacksquare$

Illustration in $\mathbb{R}^2$



Beginning $(A'x \le b') = \emptyset$

$\dim(x^{(1)}) = 1 > \dim(x^*) = 0$

Bounded LP has optimal solution

Corollary

A linear program $\max\{c^T x : x \in \mathbb{R}^n, Ax \leq b\}$ *which is feasible and bounded has an optimal solution.*

Proof: Trick: $x = x^+ - x^-$ $\quad x^+, x^- \in \mathbb{R}^n$

$x^+, x^- \geq 0$

constraint matrix

$$\begin{array}{cc} x^+ & x^- \\ \hline \end{array}$$
$$\begin{bmatrix} A & 0 \\ 0 & -A \\ -I & -I \end{bmatrix}$$

Re-WRITE: $\max\ c^T.x$, $Ax \leq b$ $\quad \Leftrightarrow \quad$ $\max\ c^T(x^+ - x^-)$, $A(x^+ - x^-) \leq b$, $x^+ \geq 0,\ x^- \geq 0$

New LP has full col. rank constraint matrix.

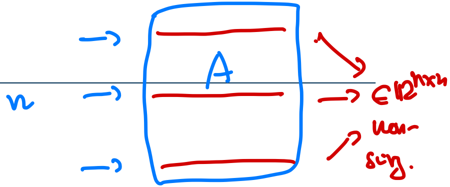
Construct

Previous thm: $\forall$ x^* feasible $\Longrightarrow$ vertex z with.

Obj. function value at least as good.

$\Rightarrow$ LP has optimal solution. $\blacksquare$

A first (inefficient) algorithm



Given: $\max\{c^T x : x \in \mathbb{R}^n, Ax \leq b\}$ with $\operatorname{rank}(A) = n$.

$A'x \leq b'$

- Initialize $M = \emptyset$
- Enumerate all sets of n row-vectors of A that are basis of $\mathbb{R}^n$
  - Solve $A'x = b'$ for corresponding sub-system $A'x \leq b'$ of $Ax \leq b$.
  - If for solution x^*: $Ax^* \leq b$ then
    $M = M + x^*$
- Output element of M with largest objective function value

Running time: $A \in \mathbb{R}^{m \times n}$ enumeration costs $\binom{m}{n}$ exponential.

A first (inefficient) algorithm

Theorem

If LP is bounded then algorithm above computes optimal solution.

We will see ...

... we can do much better.