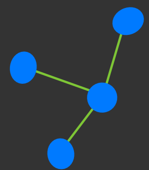
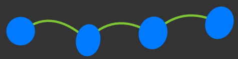
Number of unlabeled trees.

By the previous video lecture:
there are $4^2 = 16$ trees on 4 labeled vertices

What about unlabeled trees (isomorphism classes)?

(a,a) 

(a,b) 

The number of isomorphism classes is only 2.

There is no explicit formula for the number of unlabeled trees with n vertices.

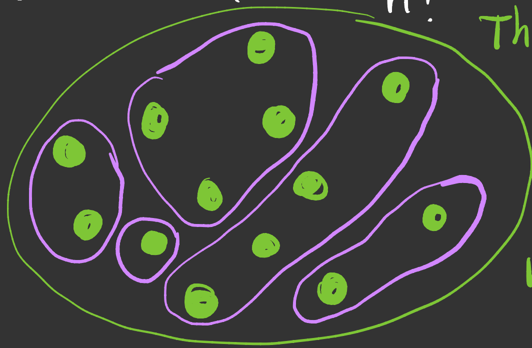
Theorem: Let T_n be the number of unlabeled trees with n vertices. Then

$$2^n \leq \frac{n^{n-2}}{n!} \leq T_n \leq 4^{n-1}$$

for $n > 30$

Proof: First we prove that $T_n \geq \dfrac{n^{n-2}}{n!}$

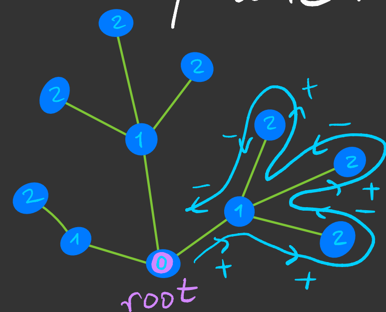
$$T_n \geq \frac{|\text{Labeled trees}|}{\text{maximal possible number of trees in one equivalence class}}$$



The number of trees in one equivalence class is $n!$

Now we show that $T^n \le 4^n$

Take an unlabeled tree T with n vertices.
Choose one vertex of T and call it a root
Embed T into plane:



We start from the root and go counterclockwise around the tree and write

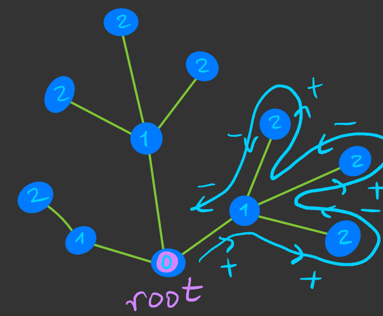
$+$ if the distance to the root increases

$-$ if the distance to the root decreases

At the end of our journey we obtain a sequence in $\{+,-\}^{2n-2}$

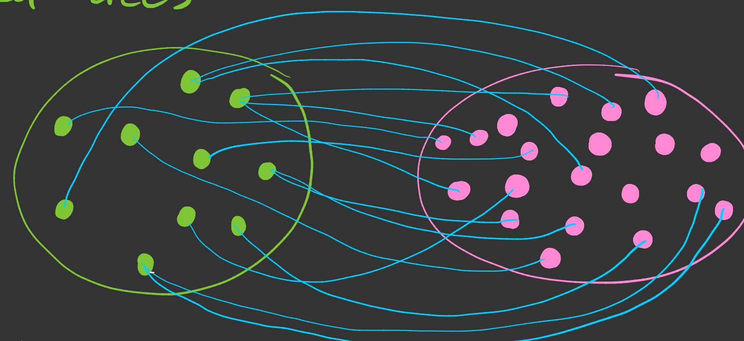
Important:

unlabeled tree can be uniquely reconstructed from the sequence.



The set of unlabeled trees

The set $\{+,-\}^{2n-2}$



- Each tree corresponds to at least one and possibly more sequences
- Each sequence corresponds to only one (unlabeled) tree
- Some sequences do not correspont to any tree

$\Rightarrow$ number of unlabeled trees $\leq$ number of sequences $= 4^{n-1}$ $\blacksquare$

Remark: A difficult result of Otter (1948) says that the number of unlabeled trees with n vertices is

$$\sim (0.5349\ldots) \cdot n^{-5/2} \cdot (2.9557\ldots)^n$$