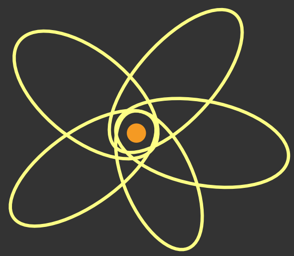
Erdős-Ko-Rado

or Sunflower 

theorem

Definition: A family $\mathcal{F}$ of sets is **intersecting** if for all $A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$.

Theorem (Erdős-Ko-Rado)

If $|X| = n$, $n \geq 2k$ and $\mathcal{F}$ is an intersecting family of k-element subsets of X, then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Remark: This Bound is tight.

It is easy to construct an intersecting family $\mathcal{F}$ of k-element subsets of cardinality $|\mathcal{F}| = \binom{n-1}{k-1}$

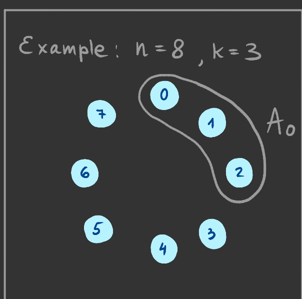
Lemma: Consider $X = \{0, 1, \dots, n-1\}$ with addition modulo n and define

$$A_s = \{s, s+1, \dots, s+k-1\} \subseteq X \quad \text{for } 0 \leq s < n.$$

Then for $n \geq 2k$ any intersecting family $\mathcal{F} \subseteq \binom{X}{k}$ contains at most k of the sets A_s

Proof of the lemma:

Without loss of generality, we may assume that $A_0 \in \mathcal{F}$.



A set A_s intersects with A_0 only for $s = 0$ or $s \in \{1, \dots, k-1, -1, \dots, -k+1\}$ mod n. We can devide these numbers into pairs

$$(j, j-k), \quad j = 1, \dots, k-1.$$

Note that $A_j \cap A_{j-k} = \emptyset$. Therefore, only one of these two sets can be contained in $\mathcal{F}$. This proves the lemma $\blacksquare$

Proof of the theorem:

We assume that $X = \{0, 1, \dots, n-1\}$ and $\mathcal{F} \subseteq \binom{X}{k}$ is an intersecting family.

For a permutation $\sigma : X \to X$, we define

$$\sigma(A_s) = \{\sigma(s), \sigma(s+1), \dots, \sigma(s+k-1)\},$$

addition again modulo n.

The lemma implies that, if we choose random s and σ independently and uniformly

(what is the underlying probability space?)

$$P[\sigma(A_s) \in \mathcal{F}] \leq \frac{k}{n}$$

This choise of $\sigma(A_s)$ is equivalent to a random choise of k-element subset of X.

Therefore
$$P(\sigma(A_s) \in \mathcal{F}) = \frac{|\mathcal{F}|}{\binom{n}{k}}.$$

Now we estimate

$$|\mathcal{F}| = \binom{n}{k} \cdot P(\sigma(A_s) \in \mathcal{F}) \leq \binom{n}{k}\frac{k}{n} = \binom{n-1}{k-1}.$$

This finishes the proof of the theorem $\blacksquare$