Random variables and their expectations

Definition: Let (Ω, P) be a finite probability space. A **random variable** on Ω is any map $f : \Omega \to \mathbb{R}$.

Example 1: Let $\left(\Omega = \{0,1\}^n,\ P(A) = \frac{|A|}{|\Omega|}\right)$ - be the probability space of n-term random sequences of 1's and 0's.

$f_1(w) :=$ number of 1s in a random sequence

$w = (w_1, \dots, w_n) \qquad w_i \in \{1, 0\}$.

Example 2: Number of fixed points in a random permutation

Example 3: Number of triangles in a random graph.

Definition: Let (Ω, P) be a finite probability space, and let f be a random variable on it.
The **expectation** of f is a real number $\mathbb{E}(f)$ defined by the formula:

$$\mathbb{E}(f) := \sum_{\omega \in \Omega} P(\{\omega\}) \cdot f(\omega).$$

Example: Let f_1 be the number of 1's in a random n-term sequence of 1's and 0's. ((Ω, P) defined on a previous slide)

$$\mathbb{E}(f_1) := \sum_{\omega \in \{0,1\}^n} f_1(\omega) \cdot 2^{-n} = 2^{-n} \sum_{k=0}^{n} k \binom{n}{k} =$$

$$= 2^{-n} \cdot \frac{d}{dx}(1+x)^n \Big|_{x=1} = 2^{-n} \cdot n \cdot 2^{n-1} = \frac{n}{2}$$

Question: Can we find an easier way to compute $\mathbb{E}(f_1)$?

Definition: Let $A \subseteq \Omega$ be an event in a probability space (Ω, P). The **indicator** of the event A is the random variable $I_A : \Omega \to \{0,1\}$ defined as

$$I_A(\omega) := \begin{cases} 1 & \text{for } \omega \in A \\ 0 & \text{for } \omega \notin A. \end{cases}$$

Lemma: For any event A, we have $\mathbb{E}(I_A) = P(A)$.

Proof: $\mathbb{E}(I_A) = \sum_{\omega \in \Omega} I_A(\omega) \cdot P(\{\omega\}) = \sum_{\omega \in A} P(\{\omega\}) = P(A)$. ∎

Theorem (linearity of expectation)

Let f, g be arbitrary random variables on a finite probability space (Ω, P) and let $\alpha \in \mathbb{R}$. Then:

$$\mathbb{E}(\alpha f) = \alpha \cdot \mathbb{E}(f) \quad \text{and} \quad \mathbb{E}(f+g) = \mathbb{E}(f) + \mathbb{E}(g).$$

Number of 1's counted again:

For $i = 1, \ldots, n$ let A_i be the event „i-th element of the random sequence is 1"

Then $P(A_i) = \frac{1}{2}$.

For each $\omega \in \{0,1\}^n$ we have

$$f_1(\omega) = I_{A_1}(\omega) + I_{A_2}(\omega) + \ldots + I_{A_n}(\omega).$$

Therefore

by linearity

$$\mathbb{E}(f_1) = \mathbb{E}(I_{A_1}) + \ldots + \mathbb{E}(I_{A_2})$$

by Lemma

$$= P(A_1) + \ldots + P(A_n)$$

$$= \frac{n}{2}.$$

Probabilistic method.

Idea: Use suitable probability spaces to prove existence results.

Theorem: Let (Ω, P) be a finite probability space and let $f: \Omega \to \mathbb{R}$ be a random variable.

If $\mathbb{E}(f) = m$ then there exists at least one elementary event ω_1 such that $f(\omega_1) \geq m$.

Analogously, there exists at least one elementary event ω_2 such that $f(\omega_2) \leq m$.

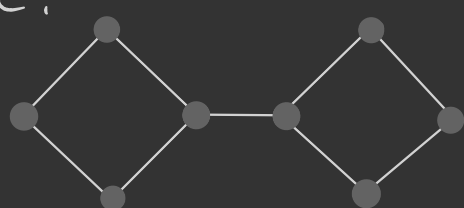
Application 1. Existence of large bipartite subgraphs.

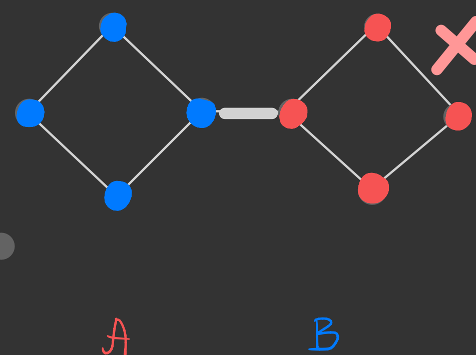
Theorem: Let G be a graph with an even number $2n$ of vertices and with $m > 0$ edges. Then the set $V = V(G)$ can be divided into two disjoint n-element subsets A and B in such a way that more then $\frac{m}{2}$ edges go between A and B.

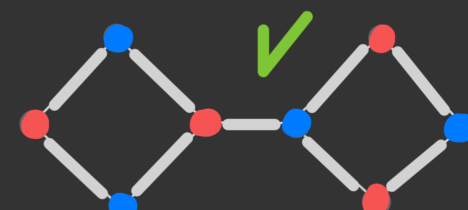
Example:

$n = 4$

$m = 9$







Proof: Consider the probability space (Ω, P) where

$$\Omega = \binom{V}{n} = \text{the set of all } n\text{-element subsets of } V(G).$$

with the probability measure

$$P(S) := \frac{|S|}{|\Omega|} \quad \text{for} \quad S \subseteq \Omega.$$

Let $A \in \Omega$ be a random n-element subset of $V(G)$, define $B := V(G) \setminus A$ its complement

Consider the following random variable:

$$X(A) := |\text{edges between } A \text{ and } B| =$$

$$|\{\{a, b\} \mid a \in A,\ b \in B,\ \{a, b\} \in E(G)\}|$$

Let us compute $\mathbb{E}(X)$.

For $e = \{u,v\} \in E(G)$ we define the event

$$C_e := \{ A \in \Omega \mid |A \cap e| = 1 \}$$

„edge e is between A and B".

Recall: The indicator function $I_{C_e}(A) = \begin{cases} 1, & A \in C_e \\ 0, & A \notin C_e. \end{cases}$

We have $X = \sum_{e \in E(G)} I_{C_e}$ and therefore

$$\mathbb{E}(X) = \sum_{e \in E(G)} \mathbb{E}(I_{C_e}) = \sum_{e \in E(G)} P(C_e).$$

We compute $P(C_e) = \dfrac{2\binom{2n-2}{n-1}}{\binom{2n}{n}} = \dfrac{n}{2n-1} > \dfrac{1}{2}$

Thus $\mathbb{E}(X) = \sum_{e \in E(G)} P(C_e) > \frac{m}{2}$. By Theorem $X(A) > \frac{m}{2}$ for some $A \in \Omega$

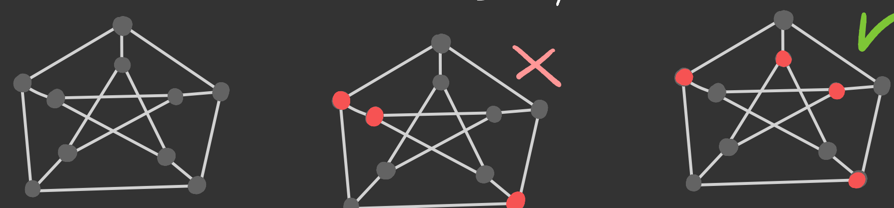
This finishes the proof $\blacksquare$

Application 2: Turán's theorem.

Definition: Let G be a graph. A set $S \subseteq V(G)$ is an *independent set* if no two vertices of S are connected by an edge.

$\alpha(G) :=$ the size of the largest independent set of vertices in the graph G.

Example:



Theorem (Turán) For every graph G we have

$$\alpha(G) \geq \frac{|V(G)|^2}{2 \cdot |E(G)| + V(G)}.$$

Lemma: For any graph G we have

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v)+1}.$$

Proof: Suppose that the vertices of G are numbered $1, \dots, n$.
Pick a random permutation π of the vertices.
Define the set $M(\pi) \subset V(G)$ by

$$M(\pi) := \{ v \in V \mid \text{all neighbours } u \text{ of } v \text{ satisfy } \pi(u) > \pi(v) \}$$

$M(\pi)$ is an independent set of G.
Therefore $|M(\pi)| \leq \alpha(G)$ for any permutation π.

$$\Rightarrow \quad \mathbb{E}(|M(\pi)|) \leq \alpha(G).$$

Now we calculate the expected size of M.
Let $v \in V$, $A_v :=$ the event "$v \in M(\pi)$."

Then $$P(A_v) = \frac{1}{\deg(v)+1}$$ why?

and $$|M(\pi)| = \sum_{v \in V} I_{A_v}(\pi).$$

Now
$$\mathbb{E}(|M|) = \sum_{v\in V} \mathbb{E}(I_{A_v}) = \sum_{v \in V} P(A_v) = \sum_{v\in V} \frac{1}{\deg(v)+1}$$

Since $\alpha(G) \geq \mathbb{E}(|M|)$ this finishes the proof of the lemma ■

Proof of Turán's theorem:

Let $|V(G)| = n$ and $d_1, \dots, d_n$ be the degrees of vertices of G.

Then $\sum_{i=1}^{n} d_i = 2 \cdot |E(G)|$

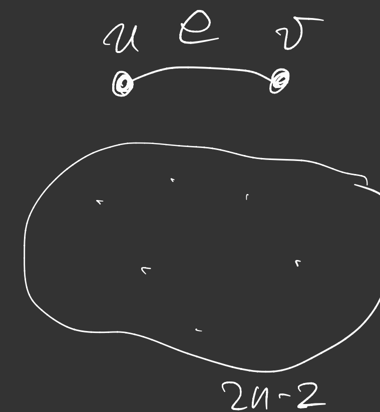
$$\sum_{i=1}^{n} \frac{1}{d_i + 1} \geq \frac{n^2}{d_1 + \dots + d_n + n} = \frac{n^2}{2 \cdot |E| + n}.$$

inequality between arithmetic mean and harmonic mean

This finishes the proof of Turán's theorem ∎

Question: Is Turán's theorem sharp? Try to construct graphs attaining Turán's bound.

Discussion



Compute $P(C_e)$

$A \subset V$ $\qquad |A| = n, \ |V| = 2n$

$e = (u, v)$

$C_e :=$ „A contains exactly one vertex of e}

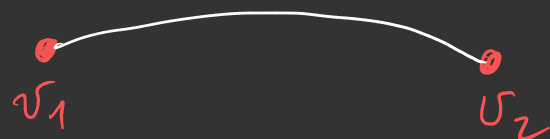
$$P(C_e) = P(A \text{ contains } u \text{ and does not contain } v) + P(A \text{ contains } v \text{ and does not contain } u)$$

$$= \frac{2\binom{2n-2}{n-1}}{\binom{2n}{n}}$$

Why is $M(\pi)$ independent set?

$M(\pi) := \{ v \in V \mid \text{all neighbours } u \text{ of } v \text{ satisfy } \pi(u) > \pi(v) \}$

Suppose:



$v_1 \in M(\pi)$

$v_2 \in M(\pi)$

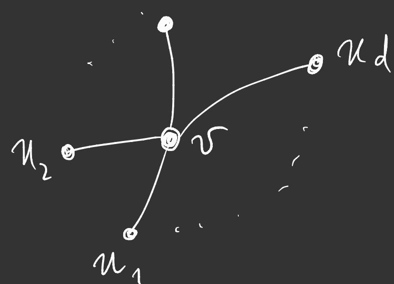
$v_1 \in M(\pi) \Rightarrow \pi(v_1) < \pi(v_2)$

$v_2 \in M(\pi) \Rightarrow \pi(v_2) < \pi(v_1)$ ↯

$\Rightarrow M$ is an independent set.

Let $v \in V$, $A_v :=$ the event "$v \in M(\pi)$."

Then $\quad P(A_v) = \dfrac{1}{\deg(v)+1}$ $\quad$ why?



$d = \deg(v)$

A_v occurs $\iff$ $\pi(v) < \pi(u_1), \ldots, \pi(v) < \pi(u_d)$

Let $K_\pi = \{k_1, k_2, \ldots, k_{d+1}\}$ be the set $\{\pi(v), \pi(u_1), \ldots, \pi(u_d)\}$

and assume $k_1 < k_2 < \ldots < k_{d+1}$

All $(d+1)$-element subsets of $\{1, 2, \ldots, n\}$ are equaly probable.

$\pi \in A_v$ if and only if $\pi(v) = k_1$

$$P(A_v) = \frac{d!}{(d+1)!} = \frac{1}{d+1}$$