Ramsey numbers

Definition: The **Ramsey number** $R(k,\ell)$ is

$R(k,\ell) = \min\{ n :$ any graph of on n vertices contains

either a clique of size k

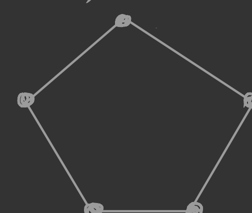
or an independent set of size $\ell \}$

Examples: $R(2,2) = ?$

G_1

G_2

$R(3,3) = ?$



$R(3,3) > 5$

Theorem (Ramsey)

The Ramsey number $R(k, \ell)$ is always finite.

Proof: We will prove this statement by induction on $k+\ell$

Base of induction: $R(2, m) = R(m, 2) = m$

Step of induction:

Lemma: $R(k, \ell) \leq R(k-1, \ell) + R(k, \ell-1)$

Proof of the lemma: Suppose that $R(k, \ell-1)$ and $R(k-1, \ell)$ are finite.
Let G be a graph on $R(k,\ell-1) + R(k-1,\ell)$ vertices.
Let v be a vertex of G. The degree of v is

either $\geq R(k-1, \ell)$

or $< R(k-1, \ell)$

If $\deg(v) \geq R(k-1, \ell)$ we consider the subraph G' of G induced on the neighbourhood of v. G' has at least $R(k-1, \ell)$ vertices.

Therefore G' contains either a clique of size $k-1$

or an i-set of size ℓ

Therefore, G contains either a clique of size k

or an i-set of size ℓ.

Why?

If $\deg(v) < R(k-1, \ell)$, then v is not connected to at least $R(k, \ell-1)$ points. We consider the subgraph G'' of G induced on the vertices not connected to v. G'' contains either a clique of size k or an i-set of size $\ell-1$.

Therefore, G contains either a clique of size k or an i-set of size ℓ. ▨

Exercise: Complete the proof of Ramsey theorem.

Corollary: $R(k, \ell) \leq \binom{k+\ell-2}{k-1}$

$R(3,3) = 6$

$R(4,4) = 18$

$R(5,5) = ?$

It is known that $43 \leq R(5,5) \leq 48$

Theorem: For any $k \geq 3$,

$$R(k,k) > 2^{\frac{k}{2}-1}.$$

Proof: It suffices to prove that there exists a graph $G(V,E)$ such that $V \geq 2^{k/2-1}$ and G contains no cliques and no independent sets of size k.

Consider the finite probability space $\mathcal{G}(n,\frac{1}{2})$ of graphs on n vertices probability of each edge is $\frac{1}{2}$.

$$P(\{G\}) = \frac{1}{|\mathcal{G}_n|} = 2^{-\binom{n}{2}}$$

For any fixed k vertices the probability that they

form a clique
form an independent set
is $2^{-\binom{k}{2}}$.

Recall: $P(A \cup B) \leq P(A) + P(B)$ for any two events A and B.

$P($A random graph in $\mathcal{G}(n, \frac{1}{2})$ contains a clique or an independent set$)$

$$= P\Big(\bigcup_{\substack{S \text{ is a} \\ \text{subset of size } k \text{ in } [n]}} \{S \text{ forms a clique}\} \cup \{S \text{ forms an i-set}\} \Big) \leq$$

$$\leq 2 \binom{n}{k} 2^{-\binom{k}{2}}$$

Suppose that $n \le 2^{\frac{k}{2}-1}$.

Then

$$2\binom{n}{k} 2^{-\frac{k(k-1)}{2}} \le 2 \cdot n^k \, 2^{-\frac{k(k-1)}{2}} \le$$

$$\le 2 \, 2^{(\frac{k}{2}-1)k} \cdot 2^{-\frac{k(k-1)}{2}}$$

$$= 2^{1+\frac{k^2}{2}-k-\frac{k^2}{2}+\frac{k}{2}} = 2^{1-\frac{k}{2}} < 1.$$

Therefore, there exists a graph with n vertices and containing neither a clique nor an i-set of size k $\blacksquare$