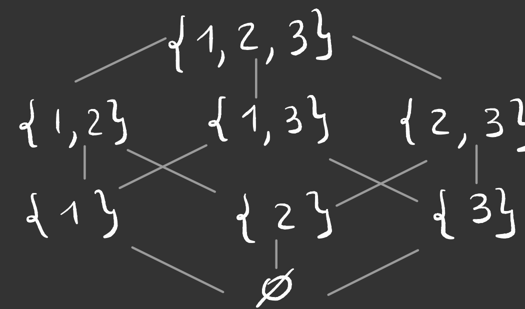
# Partialy ordered sets

# posets

# and Möbius inversion



**Definition:** A *binary relation* on a set $A$ is a subset $R \subseteq A \times A$.

A relation is *antisymmetric* provided $(a,b) \in R$ and $(b,a) \in R$ imply $a = b$.

A relation is *transitive* if $(a,b) \in R$ and $(b,c) \in R$ imply $(a,c) \in R$

A relation is *reflexive* if $(a,a) \in R$ for all $a \in R$.

**Examples:** ① The relation "$\leq$" on $\mathbb{Z}$ is antisymmetric, transitive, and reflexive

② The relation "$<$" on $\mathbb{Z}$ is antisymmetric, transitive and not reflexive

③ The relation "coprime" is not antisymmetric, not transitive, and not reflexive.

**Definition:** A *partial order* on a set $A$ is an antisymmetric, reflexive, and transitive relation $R \subseteq A \times A$.

A *partially ordered set* (or *poset* for short), is a set together with a partial order

**Example 1.** Let $A$ be a set. The set $2^A$ of subsets of $A$ is partially ordered by inclusion.

Reflexivity: $X \subseteq X$

Transitivity: if $X \subseteq Y$, $Y \subseteq Z$ then $X \subseteq Z$

Antisymmetricity: if $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$.

**Example 2.** The set $\mathbb{Z}_{\geq 1}$ is partially ordered by by the relation "$d$ divides $n$".

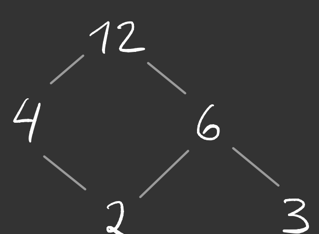
**Reflexivity:** $n$ divides $n$

**Transitivity:** if $d \mid n$ and $d' \mid d$ then $d' \mid n$

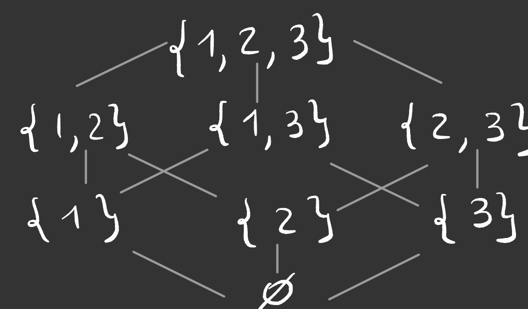
**Antisymmetricity:** if $n \mid m$ and $m \mid n$ then $m = n$.

## Hasse diagram

Let $X$ be the divisors of $12$ ordered by divisibility:




Let $Y$ be the subsets of $\{1,2,3\}$ ordered by inclusion



**Definition:** A partially ordered set $(X, \leq)$ is **locally finite** if for all $x, y \in X$ the **interval**

$$[x, y] := \{ z \in X \mid y \leq z \leq x \}$$

is finite.

We say that $0 \in X$ is a **zero element** if $0 \leq x$ for all $x \in X$.

# Möbius inversion for posets.

Let $(X, \leq)$ be a partially ordered locally finite set with 0.

Suppose that $f : X \to \mathbb{C}$ is a function

We define a new function $F : X \to \mathbb{C}$ by

$$F(x) := \sum_{y \leq x} f(y)$$

How do we recover $f$ from $F$?

$X = \mathbb{Z}_{\geq 1}$

"$\leq$" = "x divides y"

$\Downarrow$

Classical Möbius inversion formula

**Theorem** (Möbius inversion for posets)

Given a partially ordered set $X$, there is a two-variable function $M : X \times X \to \mathbb{R}$ such that

$$F(x) = \sum_{y \leq x} f(y) \iff f(x) = \sum_{y \leq x} F(y) M(y, x)$$

$M$ is called the <u>Möbius function</u> of the poset.

**Definition:** Given a partially ordered set $X$, the **incidence algebra** $\mathcal{A}(X)$ is the set of complex-valued functions $f: X^2 \to \mathbb{C}$ satisfying $f(x,y)=0$ unless $x \leq y$

$A(X)$ is a vector space over $\mathbb{C}$ with respect to pointwise addition and multiplication by scalars.
To make it into "algebra" we need one more operation:

**Definition:** Given $f, g \in \mathcal{A}(X)$, their **convolution** $f*g$ is defined by

$$f*g(x,y) = \sum_{z:\, x \leq z \leq y} f(x,z)\, g(z,y)$$

Note, that $f*g(x,y)=0$ if $x$ is not $\leq y$. Hence $f*g \in \mathcal{A}(X)$

Exercise: Convolution $*$ is associative: $(f*g)*h = f*(g*h)$.

**Definition:** The **delta function** $\delta$ is the following element of $\mathcal{A}(X)$:

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

**Exercise:** $\delta$ is the unit in the algebra $\mathcal{A}(X)$.

That is, for all $f \in \mathcal{A}(X)$

$$f * \delta = \delta * f = f.$$

Convolution $*$ is not always commutative ⚠

$$f * g \not\equiv g * f$$

Lemma: A function $f \in \mathcal{A}(X)$ has a left and right inverse with respect to the convolution if and only if $f(x,x) \neq 0$ for all $x \in X$.

Proof: Given $f \in \mathcal{A}(X)$ we find $g \in \mathcal{A}(X)$ such that

$$f * g(x,y) \overset{\text{def}}{=} \sum_{x \leq z \leq y} f(x,z)\, g(z,y) = \delta(x,y).$$

For all $x \in X$

$$f * g(x,x) = f(x,x)\, g(x,x) = \delta(x,x) = 1.$$

Therefore, the condition $f(x,x) \neq 0$ for all $x \in X$ is necessary for the existence of the inverse.

We define $g(x,x) := f(x,x)^{-1}$ for all $x \in X$.

To define $g(x,y)$ for $x < y$, we assume by induction, that we have already found all $g(z,y)$ for all $z$, satisfying $x < z \leq y$

Then

$$\delta(x,y) = 0 = \sum_{x \leq z \leq y} f(x,z)\, g(z,y)$$

$$-f(x,x)\, g(x,y) = \sum_{x < z \leq y} f(x,z)\, g(z,y)$$

We can find $g(x,y)$ from this identity because $f(x,x) \neq 0$ and we know all the terms of the finite sum on the right side.

Finally, if $f * g_1 = \delta$ and $g_2 * f = \delta$ then

$$g_2 = g_2 * \delta = g_2 * f * g_1 = (g_2 * f) * g_1 = \delta * g_1 = g_1$$

Therefore, the left and the right inverses of $f$ coinside.

This finishes the proof $\blacksquare$

**Definition:** The **zeta function** $Z(x,y)$ of the poset $(X, \leq)$ is the function

$$Z(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

The **Möbius function** $M(x,y)$ is the inverse of zeta function $Z$ with respect to convolution.

$$M * Z = Z * M = \delta$$

## Proof of the Möbius inversion for posets.

Let $f: X \to \mathbb{C}$ be a function and $F: X \to \mathbb{C}$ is defined by

$$F(x) = \sum_{y \leq x} f(y).$$

Then for a fixed $x \in X$:

$$\sum_{y \leq x} F(y) \cdot M(y,x) = \sum_{y \leq x} M(y,x) \cdot \sum_{z \leq y} f(z)$$

$$= \sum_{z,y:\, z \leq y \leq x} f(z) M(y,x) = \sum_{z \leq x} f(z) \sum_{y:\, z \leq y \leq x} M(y,x)$$

$$= \sum_{z \leq x} f(z) \Big( \sum_{z \leq y \leq x} Z(z,y) M(y,x) \Big) = \sum_{z \leq x} f(z) \big(Z * M\big)(z,x)$$

$$= \sum_{z \leq x} f(z) \cdot \delta(z,x) = f(x)$$

Now we show that the inverse is also true

Let $F: X \to \mathbb{C}$ be a function and we define $f: X \to \mathbb{C}$ by

$$f(x) = \sum_{y \leq x} F(y) M(y, x) \quad , \; x \in X.$$

Then

$$\sum_{y \leq x} f(y) = \sum_{y \leq x} \sum_{z \leq y} F(z) M(z, y) =$$

$$= \sum_{z, y \,:\, z \leq y \leq x} F(z) M(z, y) = \sum_{z, y \,:\, z \leq y \leq x} F(z) M(z, y) Z(y, x)$$

$$= \sum_{z \,:\, z \leq x} F(z) \Big( \sum_{y \,:\, z \leq y \leq x} M(z, y) Z(y, x) \Big) = \sum_{z \leq x} F(z) \delta(z, x) = F(x)$$

$\blacksquare$

Now let as compute the Möbius function for the following two posets

- The set $2^A$ ordered by inclusion, where $A$ is a finite set
- The set $\mathbb{Z}_{\geq 1}$ ordered by "$d$ is a divisor of $n$"

Lemma: Let $2^A$ be the set of subsets of a finite set $A$. $2^A$ is partially ordered by inclusion.
The Möbius function of $2^A$ is given by

$$M(x,y) = (-1)^{|x|-|y|}$$

where $x \subseteq y \subseteq A$.

Proof: Next slide

We have to show that $M * Z = \delta$

Let $x, y$ be two subsets of $A$ such that $x \subseteq y$

We compute

$$M * Z(x,y) = \sum_{z:\, x \subseteq z \subseteq y} M(x,z)\, Z(z,y) =$$

$$= \sum_{z:\, x \subseteq z \subseteq y} (-1)^{|x| - |z|} = $$

$$= \sum_{w \subseteq y \setminus x} (-1)^{|w|} = \begin{cases} 0, & |y \setminus x| \geq 1 \\ 1, & |y \setminus x| = \emptyset \end{cases} = \delta(x,y)$$

We define $w := z \setminus x$ new summation variable $w$ runs over all subsets of $y \setminus x$. Moreover, $w \longmapsto x \cup w =: z$ is a bijection between subsets of $y \setminus x$ and the set of subsets $z$ such that $x \subseteq z \subseteq y$.

This finishes the proof $\blacksquare$

Lemma: Consider the set $\mathbb{Z}_{\geq 1}$ partially ordered by the relation " $x$ divides $y$ ".

Then the Möbius function of this poset is

$$M(x,y) = \mu(y/x).$$

" $\leq$ " = "divides"

Proof: We need to show that $M * Z = \delta$.

For $x, y \in \mathbb{Z}_{\geq 1}$ with $x \mid y$ we

$$M * Z(x,y) = \sum_{\substack{z \in \mathbb{Z}_{\geq 1} \\ x \leq z \leq y}} M(x,z) \cdot Z(z,y) =$$

We have evaluated this sum in the first video

$$= \sum_{z:\, x \mid z,\, z \mid y} \mu\left(\frac{z}{x}\right) = \left\{\begin{array}{l}\text{we make a change} \\ \text{the summation variable} \\ d := \frac{z}{x} \\ d \text{ runs over divisors of } \frac{y}{x}\end{array}\right\} = \sum_{d \mid \frac{y}{x}} \mu(d) = \begin{cases} 1, & \frac{y}{x} = 1 \\ 0 & \text{otherwise} \end{cases} = \delta(x,y)$$

$\blacksquare$