Matrix tree theorem

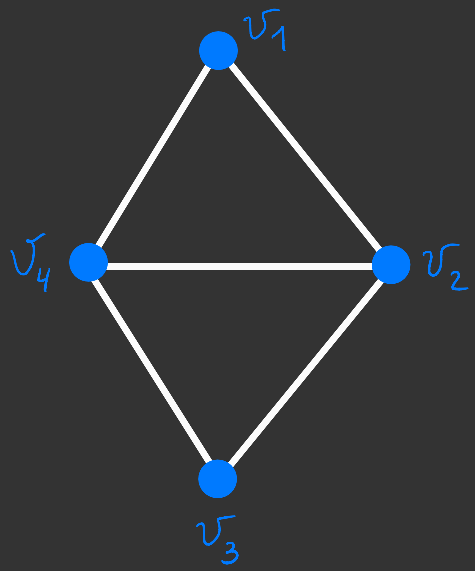
or

Kirchhoff's theorem

Theorem (Kirchhoff) Let G be a connected graph on n vertices. Then the rank of the Laplace matrix $L(G)$ is $n-1$.
Let $0, \lambda_1, \dots, \lambda_{n-1}$ be the eigenvalues of $L(G)$. Then the number of spanning trees of G is

$$\frac{1}{n} \lambda_1 \cdot \lambda_2 \cdots \lambda_{n-1}.$$

Example:



$$L(G) = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

Characteristic poly of this matrix:

$$\det(L - x \cdot id_4) = x^4 - 10x^3 + 32x^2 - 32x$$

Eigenvalues of this matrix are:

$$0, 2, 4, 4$$

Matrix tree theorem implies: this graph has 8 spanning trees.

Let $M = M(G)$ be an incidence matrix of G.

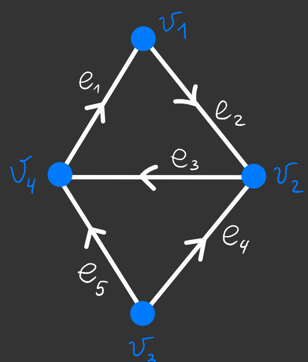
Let $M_0(G)$ be the matrix obtained from $M(G)$ by removing the last row. $\left(M_0(G) \in M_{(|V|-1)\times|E|}(\mathbb{Z})\right)$

Let $S \subset E$ be a subset of edges such that
$$|S| = |V(G)| - 1.$$

$M_0(S) :=$ submatrix of $M_0(G)$ formed by columns of M_0 indexed by edges of S.

Lemma: Let S be a set of $n-1$ edges of G. If S does not form the set of edges of a spanning tree, then $\det(M_0(S)) = 0$. If S is the set of edges of a spanning tree of G, then $\det(M_0(S)) = \pm 1$.

Example:



$G, \mathcal{O}$

$$M(G,\mathcal{O}) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & -1 \end{bmatrix}$$

$S = \{e_1, e_3, e_5\}$

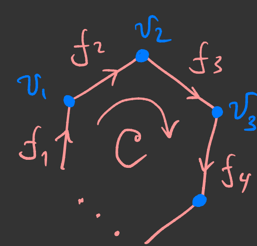
$$M_0(S) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & -1 \end{bmatrix}$$

$\det(M_0(S)) = -1$

Proof of the lemma:

First, suppose that S is <u>not</u> the set of edges of a spanning tree. Then some subset R of S forms the edges of a cycle C in G. Suppose that the cycle C has edges $f_1, \dots, f_s$ in this order. Let $\vec{w}_1, \dots, \vec{w}_s$ be the corresponding column vectors of $M_0(S)$.

Define $$k_i := \begin{cases} +1 & \text{if orientation of } f_i \text{ coincides with orientation of } C. \\ -1 & \text{if not.} \end{cases}$$



$$\begin{array}{c|cccc} & f_1 & f_2 & f_3 & \cdots \\ \hline v_1 & -1 & +1 & 0 & \\ v_2 & 0 & -1 & +1 & \\ v_3 & \vdots & 0 & -1 & \\ \vdots & 0 & \vdots & 0 & \\ v_s & +1 & 0 & \vdots & \\ \hline & \vec{w}_1 & \vec{w}_2 & \vec{w}_3 & \cdots \end{array}$$

We have: $$\sum_{i=1}^{s} k_i \cdot \vec{w}_i = 0.$$

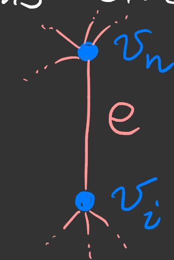
Therefore $\operatorname{rk}(M_0(S)) < n-1 \implies \det(M_0(S)) = 0.$

Now suppose that S is the set of edges of a spanning tree T.

Recall: v_n is the last vertex of G which corresponds to the row removed from M to obtain M_0.

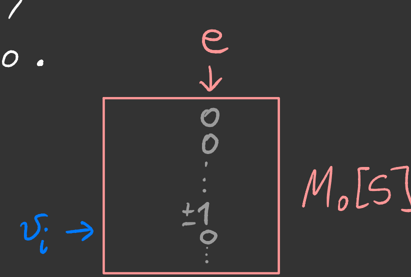
Let e be an edge of T which is connected to v_n.

The column of $M_0(S)$ indexed by e contains exactly one non-zero entry (which is ± 1).



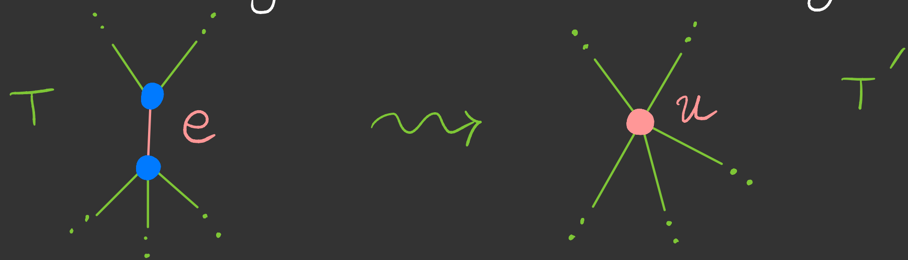
Remove from $M_0(S)$ the row containing this non-zero entry (the row corresp. to v_i) and the column corresponding to e.

We obtain a $(n-2)\times(n-2)$ matrix M_0'.



We have $\det(M_0(S)) = \pm \det(M_0')$

Let T' be the tree obtained from T by contracting the edge e to a single vertex u:



Then M_0' is the matrix obtained from the incidence matrix of T' by removing the row indexed by u.

By induction on the number n of vertices we have $\det(M_0') = \pm 1$ (the case $n=2$ is trivial).

This finishes the proof of the lemma. ∎

Let A be a rectangular matrix of size $m \times n$.
Suppose $m \le n$ and S is an m-element subset of $\{1, 2, \dots, n\}$.
Then we denote by $A[S]$ the matrix $(A_{i,j})_{\substack{i=1,\dots,m \\ j \in S}}$ consisting of columns of A indexed by elements of S.

Example.

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 5 & 6 & 7 \end{pmatrix}, \quad S = \{1, 3\}, \quad A[S] = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}.$$

Theorem (Binet-Cauchy)

Let $A, B \in M_{m\times n}(\mathbb{C})$. If $m \le n$ then

$$\det(A \cdot B^t) = \sum_S (\det A[S])(\det B[S])$$

where S runs over all m-element subsets of $\{1, 2, \dots, n\}$.

Proof of Kirchhoff's theorem:

Recall: $L(G)$ is the Laplace matrix of G.

Let $L_0(G)$ be the matrix obtained from $L(G)$ by removing the last row and the last column.

$$L(G) = \left[\begin{array}{c|c} L_0(G) & \begin{matrix} L_{1,n} \\ L_{2,n} \\ \vdots \end{matrix} \\ \hline \begin{matrix} L_{1,n} & L_{2,n} & \cdots \end{matrix} & \end{array}\right] \qquad L_{in} = -\sum_{j=1}^{n-1} L_{ij}, \quad i=1,\dots,n \qquad \operatorname{rk} L(G) < n$$

$$\left[\begin{array}{c|c} L_0 & \begin{matrix} 0 \\ 0 \\ \vdots \end{matrix} \\ \hline \begin{matrix} 0 & 0 & \cdots \end{matrix} & 0 \end{array}\right] \cdot \left[\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{matrix} & \begin{matrix} -1 \\ -1 \\ \vdots \\ -1 \end{matrix} \\ \hline \begin{matrix} 0 & 0 & \cdots \end{matrix} & 1 \end{array}\right] = \left[\begin{array}{c|c} L_0 & \begin{matrix} L_{1,n} \\ L_{2,n} \\ \vdots \end{matrix} \\ \hline \begin{matrix} 0 & 0 & \cdots \end{matrix} & 0 \end{array}\right]$$

$$\left[\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \begin{matrix} -1 & -1 & \cdots & -1 \end{matrix} & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} L_0 & \begin{matrix} 0 \\ 0 \\ \vdots \end{matrix} \\ \hline \begin{matrix} 0 & 0 & \cdots \end{matrix} & 0 \end{array}\right] \cdot \left[\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{matrix} & \begin{matrix} -1 \\ -1 \\ \vdots \\ -1 \end{matrix} \\ \hline \begin{matrix} 0 & 0 & \cdots \end{matrix} & 1 \end{array}\right] = \left[\begin{array}{c|c} L_0 & \begin{matrix} L_{1,n} \\ L_{2,n} \\ \vdots \end{matrix} \\ \hline \begin{matrix} L_{n,1} & L_{n,2} & \cdots \end{matrix} & L_{n,n} \end{array}\right]$$

Exerise: Show that $\det(L_0(G)) = \frac{1}{n} \lambda_1 \cdot \lambda_2 \cdot \cdots \lambda_{n-1}$.

We have $L_0 = M_0 \cdot M_0^t$.

By Binet-Cauchy theorem

$$\det(L_0) = \sum_{\substack{S \subset E \\ |S| = n-1}} (\det M_0[S]) (\det M_0^t[S])$$

$$= \sum_{\substack{S \subset E \\ |S| = n-1}} (\det M_0[S])^2 =$$

Here we use the Lemma

$$= \sum_{S} (\pm 1)^2 + \sum_{S} (0)^2$$

S is the set of edges of a spanning tree

S is not the set of edges of a spanning tree

= the number of spanning trees in G. ∎