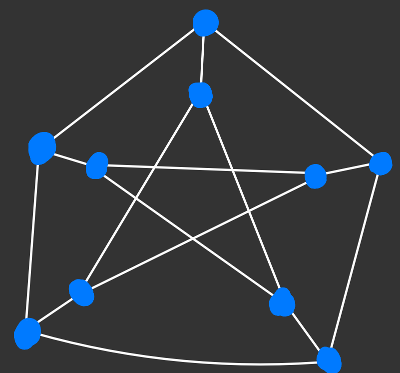
Graph theory

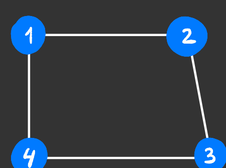


Basic definitions

Definition: A *graph* G is an ordered pair (V, E) where V is a set of elements called *vertices* and E is a set of 2-element subsets of V, called *edges*.

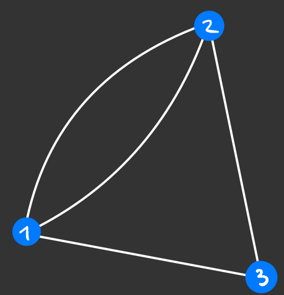
Example: $V = \{1, 2, 3, 4\}$

$E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}\} \subset \binom{V}{2}$

$G =$ 

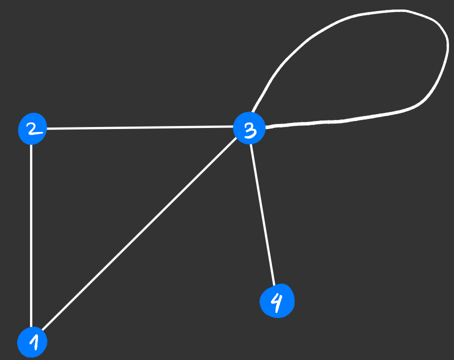
Terminology:
undirected simple graph

Non-examples:



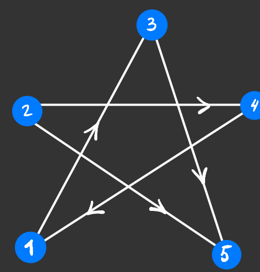
Multiple edges

undirected multigraph



Loops

multiple/simple undirected graph with loops



Directions of edges

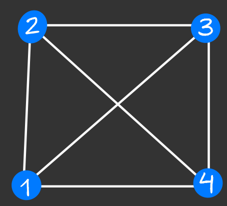
directed graph

Important graphs

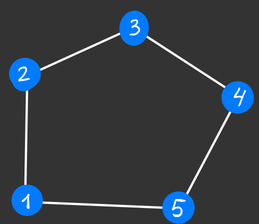
Definition: Let V be a finite set.
A **complete graph** on vertices V (or a **clique**) is the graph $G = \left(V, \binom{V}{2}\right)$

A complete graph with n vertices is denoted K_n

Example: K_4



C_5



Definition: The **cycle** C_n

$V = \{1, 2, \dots, n\}$ $\quad E = \{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}, \{n, 1\}\}$

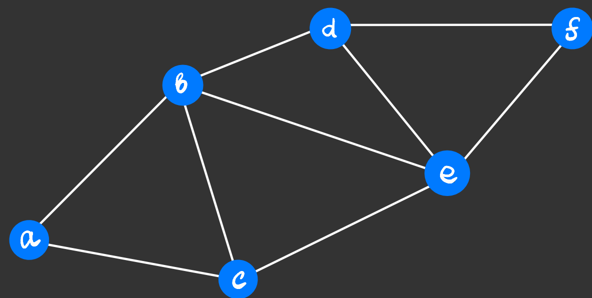
Let $G = (V, E)$ be a graph

Definition: Let $v_1, v_2 \in V$ be vertices of G.
If $\{v_1, v_2\} \in E$ we say that v_1 and v_2 are **connected by an edge** or **adjacent**

We also say that the edge $\{v_1, v_2\}$ is **adjacent** to vertices v_1 and v_2

Definition: A **degree** of a vertex $v \in V$ is the number of edges adjaicent to it

Example



$\deg(a) = 2$

$\deg(b) = 4$

$\deg(c) = 3$

The hand shake lemma: The sum of degrees of all vertices in a (finite) graph G is an even number.

$2+4+3+3+4+2 = 18.$

One line proof: Sum of degrees of all vertices is equal to twice the number of edges. $\blacksquare$

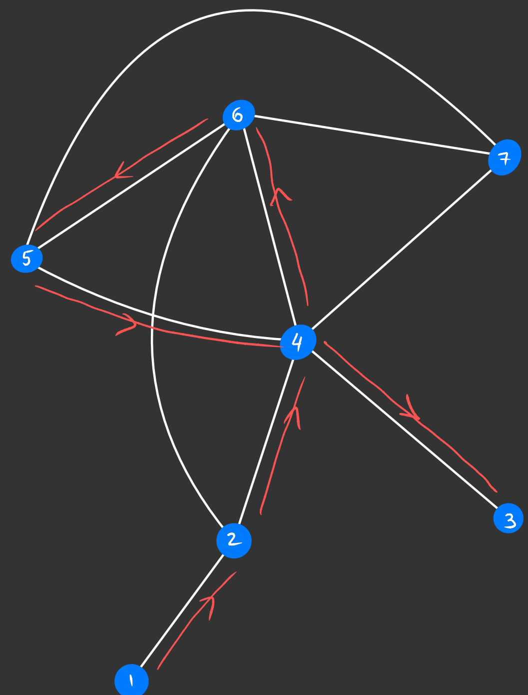
Walks and pathes

Definition: A *walk* is a graph G is a sequence of nodes $v_0, v_1, \ldots, v_k$ such that v_0 is adjacent to v_1, v_1 is adjacent to v_2, v_2 is adjacent to v_3, etc; any two consecutive nodes in the sequence must be connected by an edge.

A *path* is a walk such that all its vertices are distinct.

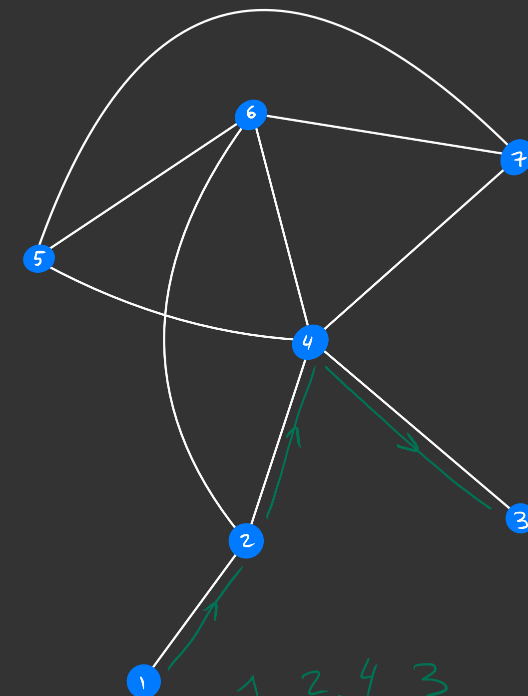
A *closed walk* is a walk such that the first vertex v_0 coinsides with the last one v_k.

Examples



1, 2, 4, 6, 5, 4, 3

A walk

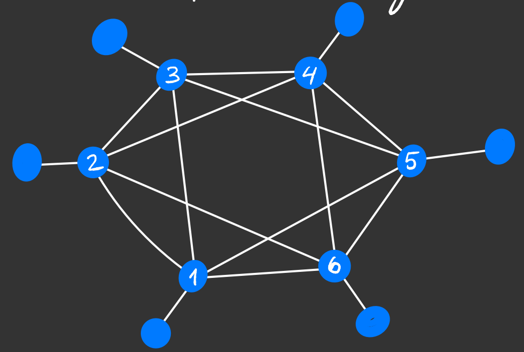


1, 2, 4, 3

A path

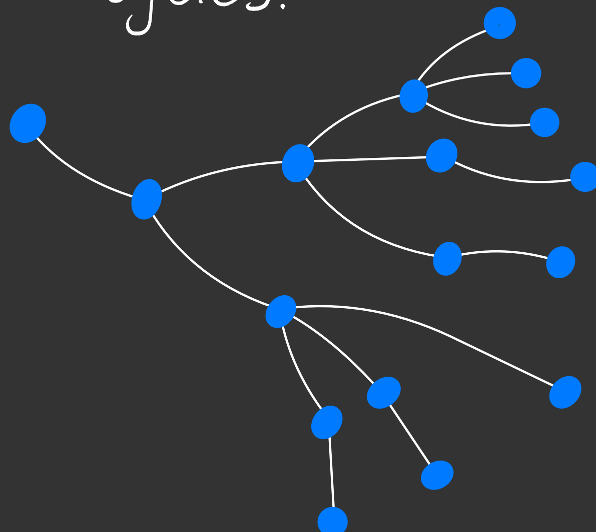
Definition: A graph $G=(V,E)$ is **connected** if for every two vertices $u,v \in V$ there exists a path in G between them.

Definition: A **cycle** in a graph $G=(V,E)$ is a sequence of distinct vertices $v_1, \dots, v_r \in V$ with $r \geq 3$ such that v_1 is adjacent to v_2, v_2 is adjacent to $v_3, \dots, v_{r-1}$ is adjacent to v_r and moreove v_r is adjacent to v_1



1, 2, 3, 4, 5, 6 is a cycle

Definition: A **tree** is a connected graph without cycles.



A tree

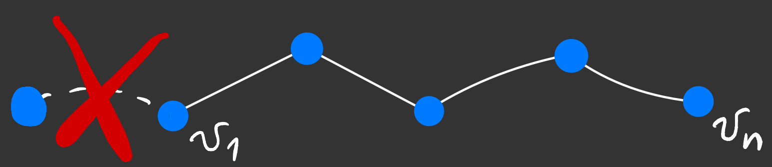
Definition: A vertex of degree 1 in a tree is called a **leaf**.

Lemma: Every (finite) tree on $n \geq 2$ vertices has at least two leaves.

Proof: Consider a path of maximum length, say $v_1, v_2, \ldots, v_n$.

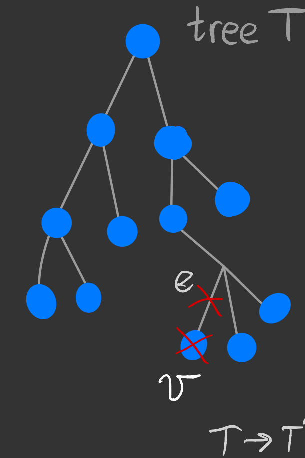
Tree is connected, therefore such path exists and has length at least 2.

The initial point v_1 and the final point v_n must be leaves, otherwise the path can be extended.



This finishes the proof ∎

Lemma: Every tree on n vertices has exactly $n-1$ edges.



Proof. We prove the lemma by induction on n.

Base of induction: $n=1 \Rightarrow 0$ edges.

Step of induction: Suppose the lemma is true for all trees with $\leq n$ vertices.

Let $T = (V, E)$ be a tree with $n+1$ vertices

By the previous lemma T has a leave, say $v \in V$.

Let e be the unique edge adjaicent to v.

We define a new graph $T' := (V \setminus \{v\}, E \setminus \{e\})$.

New graph T' contains no cycles and is connected $\Rightarrow T'$ is a tree.

By induction hypothesis $|E \setminus \{e\}| = |V \setminus \{v\}| - 1 = n-1$. Therefore $|E| = |V| - 1$ ▨