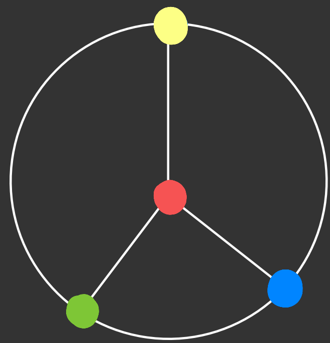
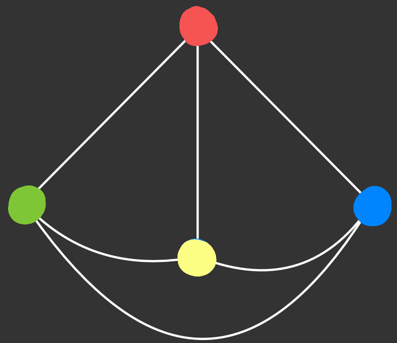
Graph isomorphisms





Graph isomorphisms

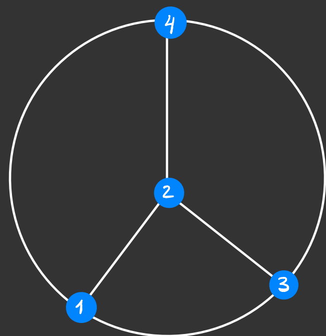
Two graphs $G = (V, E)$ and $G' = (V', E')$ are considered identical if they have the same set of vertices and the same set of edges, that is $V = V'$ and $E = E'$

Definition: Two graphs $G = (V, E)$ and $G' = (V', E')$ are called isomorphic if there is a bijection $f : V \xrightarrow{\sim} V'$ such that for all $x, y \in V$:

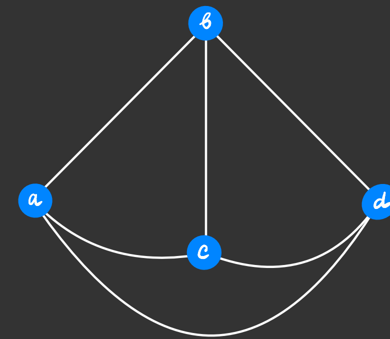
$$\{x, y\} \in E \quad \text{if and only if} \quad \{f(x), f(y)\} \in E'.$$

Such an f is called an isomorphism of the graphs G and G'. Notation: $G \simeq G'$

Example



$\simeq$



isomorphism

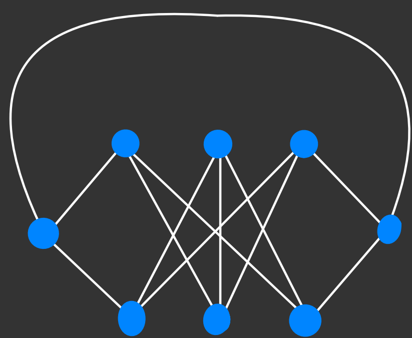
$1 \longrightarrow a$

$2 \longrightarrow b$

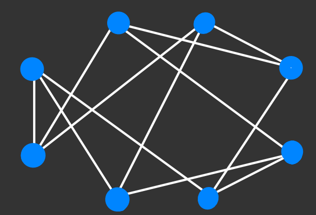
$3 \longrightarrow d$

$4 \longrightarrow c$

Remark: In general, deciding whether two graphs are isomorphic is a difficult computational problem.
Finding efficient algorithms is an active research area.



$$\overset{?}{\cong}$$

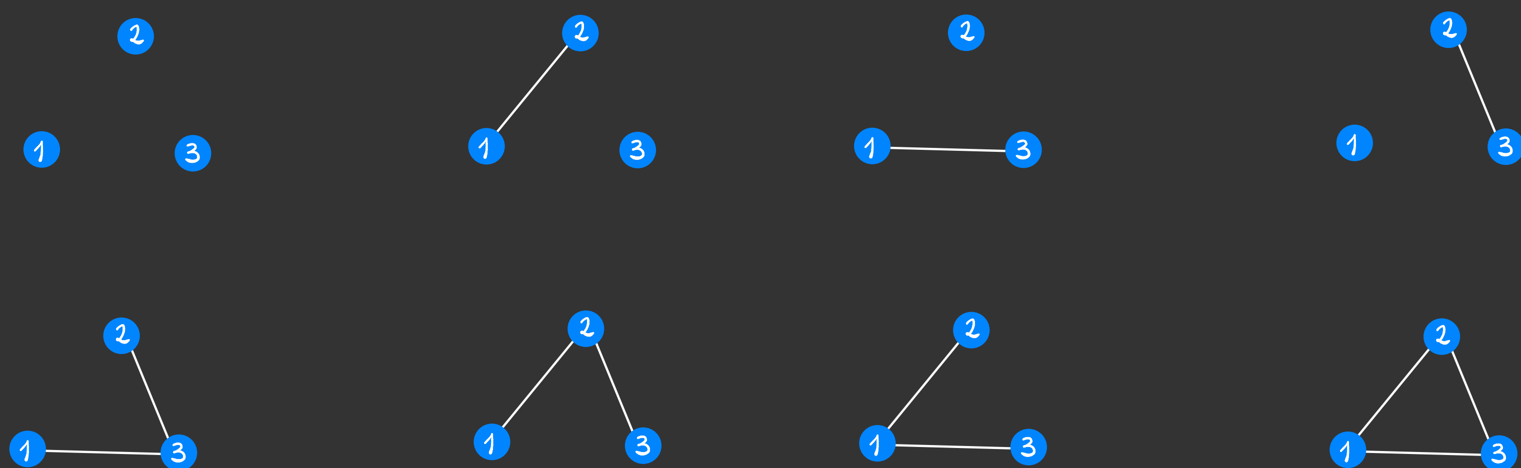


Number of isomorphism classes of graphs

Fix the vertices $\{1, 2, 3\}$

Q: How many different graphs on this vertices are there?

A: $2^{\binom{3}{2}} = 8$

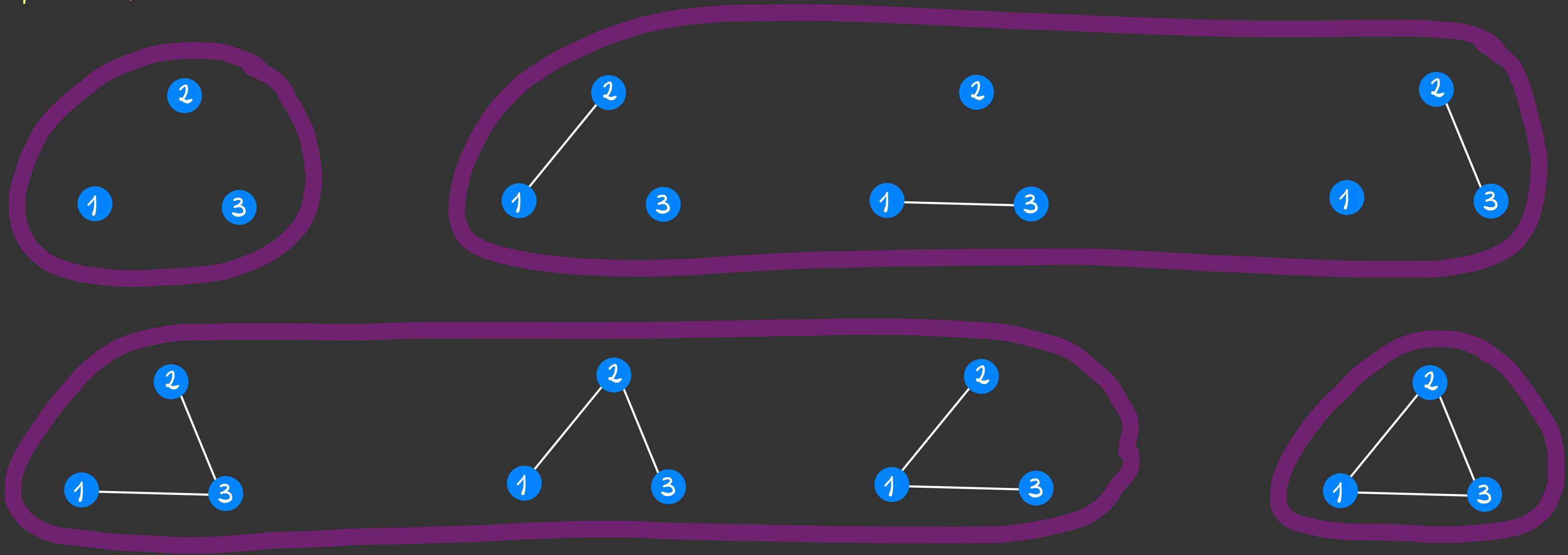


Number of isomorphism classes of graphs

Fix the vertices $\{1, 2, 3\}$

Q: How many isomorphism classes of graphs on this vertices are there?

A: 4



What is the number of isomorphism classes of graphs with vertices $\{1, 2, \dots, n\}$

The number of different graphs is $2^{\binom{n}{2}}$.

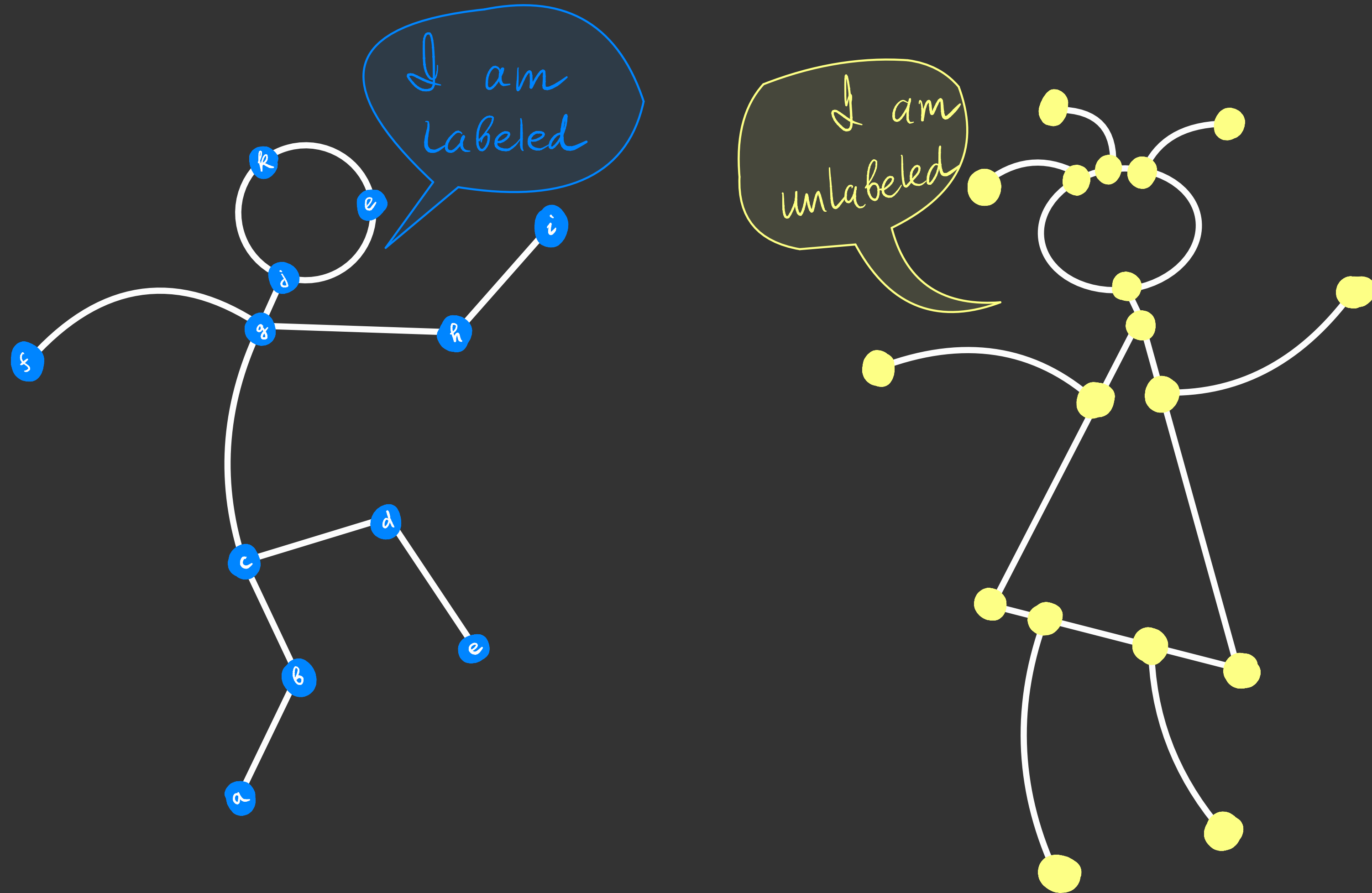
Each isomorphism class contains at most $n!$ elements

($n!$ is the number of bijections from $\{1, \dots, n\}$ to $\{1, \dots, n\}$).

Therefore

$$\frac{2^{\binom{n}{2}}}{n!} \leq \left| \text{isomorphism classes} \right| \leq 2^{\binom{n}{2}}$$

Notation: An isomorphis class of graphs is also called an unlabeled graph.


I am labeled
I am unlabeled