Combinatorial probability

⚠ Disclamer:

This is <u>not</u> an introduction to probability theory

The goal of this lecture is to emphasize the importance of combinatorial counting and to put it into a new context.

Definition: *A finite probability space* is a pair (Ω, P) where Ω is a finite set and $P: 2^{\Omega} \to [0,1]$ is a function assigning a number from the interval $[0,1]$ to every subset of Ω such that:

(1) $P(\emptyset) = 0$

(2) $P(\Omega) = 1$

(3) $P(A \cup B) = P(A) + P(B)$ for any two disjoint sets $A, B \subseteq \Omega$.

Probability theory to set theory dictionary.

The set Ω can be thought as the set of <u>all</u> possible outcomes of some random experiment.

Elements of Ω are called elementary events.

Subsets of Ω are called events.

⚠ "Elementary events" are not "events".

Let $\omega \in \Omega$, $A, B \subseteq \Omega$

$\omega \in A \iff$ event A occured

$\omega \in A \cap B \iff$ both events A and B occured

$A \cap B = \emptyset \iff$ events A and B are incompatible

$P(A) \iff$ the probability of event A

Examples of finite probability spaces.

① A random sequence of 0s and 1s

$\Omega = \{0,1\}^n$ = all n-term sequences of 0s and 1s

$|\Omega| = 2^n$

For $A \subseteq \Omega$ $\quad P(A) := \dfrac{|A|}{|\Omega|} = \dfrac{|A|}{2^n}$

Consider a set $A \subseteq \Omega$

$A := \{(\omega_1, \dots, \omega_n) \mid \omega_1 = 1\}$

A is the event:

„the first element of a random sequence is 1."

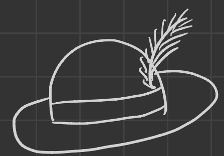
Q: What is a probability of A?

② A random permutation

$\Omega = S_n$ = set of all permutations of the set $\{1, \dots, n\}$

For $A \subset \Omega$ $\qquad P(A) = \frac{|A|}{n!}$

<u>Recall</u>: Hatcheck lady problem



③ A random graph

⚠ There are many ways to define random graph.
This is only one version:

$\Omega = \mathcal{G}_n :=$ set of all possible (labeled) graphs on vertex set $V = \{1, \dots, n\}$.

for $A \subset \mathcal{G}_n$ $\qquad P(A) := \frac{|A|}{|\Omega|} = |A| \cdot 2^{-\binom{n}{2}}$.

Proposition: A random graph is almost never a tree, i. e.

$$\lim_{n\to\infty} P\Big(\underbrace{\text{"A graph in } \mathcal{G}_n \text{ is a tree"}}_{=:\, T_n \subset \mathcal{G}_n}\Big) = 0 .$$

Proof:

$$P(T_n) = \frac{|T_n|}{|\mathcal{G}_n|}$$

By Caley's theorem $|T_n| = n^{n-2}$

$$\lim_{n\to\infty} \frac{n^{n-2}}{2^{\frac{n(n-1)}{2}}} = \lim_{n\to\infty} e^{-\ln(2)\frac{n(n-1)}{2} + (n-2)\ln(n)} = 0$$

$\blacksquare$

Definition: Two events A, B in probability space (Ω, P) are called **independent** if

$$P(A \cap B) = P(A) \cdot P(B).$$

Example: Let $\Omega = \{0,1\}^n$, the prob. space of random sequences

Consider 2 events:

event A := „the first element of the sequence is 1"

event B := „the second element of the sequence is 1"

$$A = \{(w_1, \dots, w_n) \mid w_1 = 1\} \subset \Omega$$

$$B = \{(w_1, \dots, w_n) \mid w_2 = 1\} \subset \Omega$$

Events A and B are independent.

Definition: Events $A_1, \dots, A_n \subseteq \Omega$ are independent if for each set of indices $I \subseteq \{1, \dots, n\}$

$$P\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i).$$