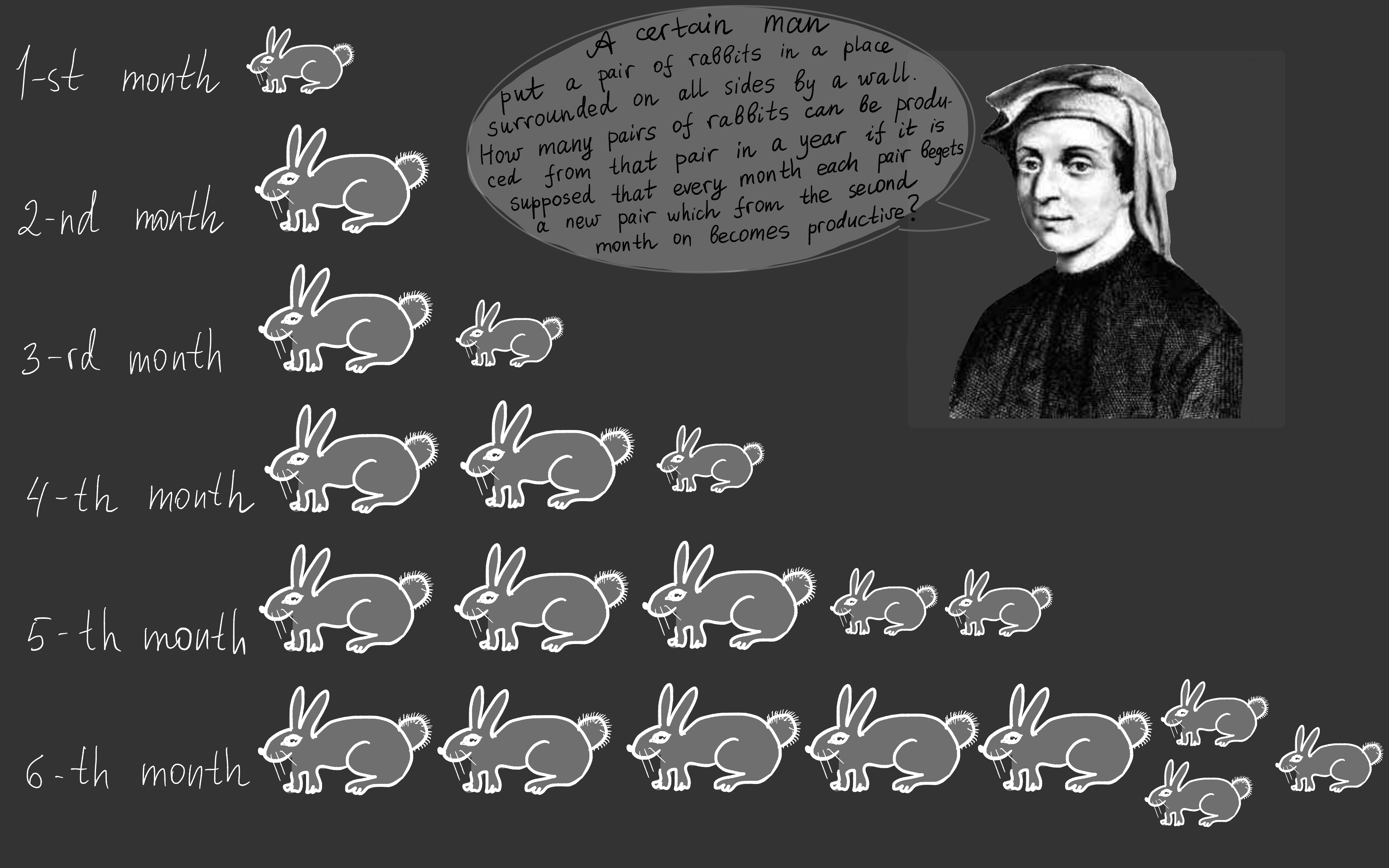
Fibonacci numbers and the golden ratio


1-st month
2-nd month
3-rd month
4-th month
5-th month
6-th month
A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

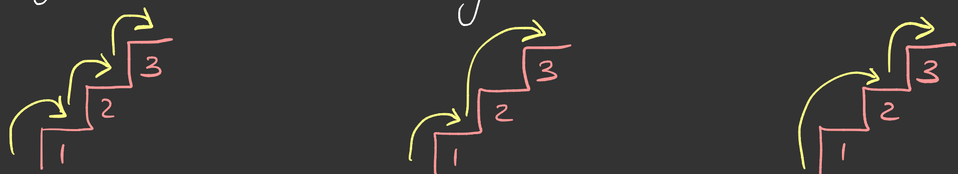
Definition: The Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined by the following recursive formula:

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}, \quad n \geq 1.$$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34 ...

Another intepretation of Fibonacci numbers:

Let S_n denote the number of ways one can climb n stairs if allowed to jumb 1 or 2 stairs at once



S_n is the number of solutions of the equation $x_1 + x_2 + .. + x_k = n$ where $x_i \in \{1, 2\}$ and k is not fixed.

We observe that $S_1 = 1$ $S_2 = 2$ and $S_n = S_{n-1} + S_{n-2}$. Thus $S_n = F_{n+1}$

Identities for Fibonacci numbers

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

1) $0+1=1$

$0+1+1=2$

$0+1+1+2=4$

$0+1+1+2+3=7$

$$F_0+F_1+F_2+\dots+F_n=$$

2) 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

$$F_{2n+1}=$$

3) $1=1$

$1+2=3$

$1+2+5=8$

$1+2+5+13=21$

$$F_1+F_3+F_5+\dots+F_{2n+1}=$$

Fibonacci numbers and golden ratio

Golden ratio is the number

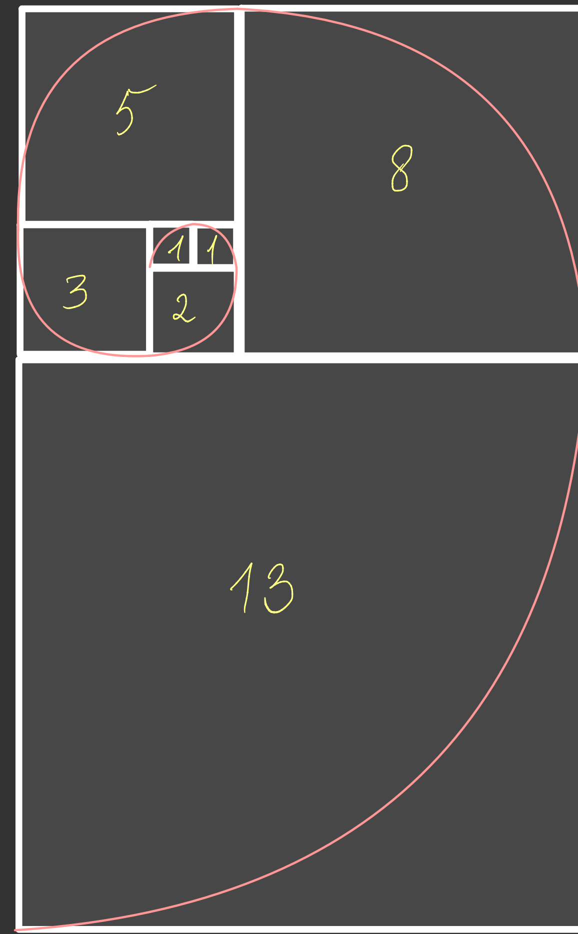
$$\varphi = \frac{1+\sqrt{5}}{2} = 1.618033\ldots$$

$$\frac{F_2}{F_1} = \frac{1}{1} = 1, \quad \frac{F_3}{F_2} = \frac{2}{1} = 2$$

$$\frac{F_4}{F_3} = \frac{3}{2} = 1.5, \quad \frac{F_5}{F_4} = \frac{5}{3} = 1.66\ldots$$

$$\frac{F_6}{F_5} = \frac{8}{5} = 1.6 \qquad \frac{F_7}{F_6} = \frac{13}{8} = 1.625$$

"Conjecture": $\lim\limits_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi$



Explicit formula for Fibonacci numbers

Consider the generating function of the Fibonacci sequence:

$$F(x) = F_0 + F_1 x + F_2 x^2 + \dots + F_n x^n + \dots$$

$$x^2 F(x) = \sum_{n=2}^{\infty} F_{n-2} x^n$$

$$x \cdot F(x) = \sum_{n=1}^{\infty} F_{n-1} x^n$$

$$F(x) - x \cdot F(x) - x^2 F(x) = \sum_{n=0}^{\infty} F_n x^n - \sum_{n=1}^{\infty} F_{n-1} x^n - \sum_{n=2}^{\infty} F_{n-2} x^n$$

$$= F_0 + x F_1 - x \cdot F_0 + \sum_{n=2}^{\infty} (F_n - F_{n-1} - F_{n-2}) x^n = 0 + x - 0 = x$$

Thus we find:

$$F(x)\cdot(1-x-x^2)=x$$

Thus

$$F(x)=\frac{x}{1-x-x^2}$$ as formal power series.

Exercise: Show that there exists a unique formal power series $f(x)=\sum_{n=0}^{\infty} f_n x^n$ such that

$$f(x)\cdot(1-x-x^2)=x$$

The function $\frac{x}{1-x-x^2}$ has derivatives of all orders at the point $x=0$.

We will compute the Taylor expansion of this function at $x=0$.

The polynomial $1-x-x^2$ has two roots:

$$x_1 = \frac{-1-\sqrt{5}}{2} \qquad x_2 = \frac{-1+\sqrt{5}}{2}$$

$$\frac{x}{1-x-x^2} = \frac{-x}{(x-x_1)(x-x_2)} = \left(\frac{1}{x-x_1} - \frac{1}{x-x_2}\right)\frac{x}{x_2-x_1}$$

$$= \frac{x}{\sqrt{5}}\left(\frac{-1}{x_1}\cdot\frac{1}{1-\frac{x}{x_1}} + \frac{1}{x_2}\cdot\frac{1}{1-\frac{x}{x_2}}\right)$$

$$= \frac{x}{\sqrt{5}}\left(\sum_{n=0}^{\infty} -\frac{1}{x_1}\cdot\left(\frac{x}{x_1}\right)^n + \sum_{n=0}^{\infty} \frac{1}{x_2}\left(\frac{x}{x_2}\right)^n\right)$$

$$= \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\left(-x_1^{-n-1} + x_2^{-n-1}\right)x^{n+1} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)x^n$$

Therefore the Fibonacci numbers satisfy:

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$$= \frac{1}{\sqrt{5}}\left(\varphi^n - \left(-\frac{1}{\varphi}\right)^n\right), \quad n \geq 0$$

$$F_0 = \frac{1}{\sqrt{5}}(1 - 1) = 0 \checkmark$$

$$F_1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 1 \checkmark$$