# Möbius inversion formula

Let $f : \mathbb{Z}_{\geq 1} \to \mathbb{C}$ be a function

We define a new function $F : \mathbb{Z}_{\geq 1} \to \mathbb{C}$ by

$$F(n) := \sum_{d|n} f(d) \quad , \quad n \in \mathbb{Z}_{\geq 1}.$$

the summation is taken over all positive divisors of $n$.

Example: Let $f(n) = 1$ for all $n \in \mathbb{Z}_{\geq 1}$

Then $F(n) = \sum_{d|n} 1 =$ number of divisors of $n$

Question: Suppose that we know $F$.

How do we recover $f$?

**Definition:** The <u>Möbius function</u> $\mu : \mathbb{Z}_{\geq 1} \to \mathbb{Z}$ is defined as follows:

Suppose that $n \in \mathbb{Z}_{\geq 1}$ has the prime factorization

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_r^{e_r}$$

then

$$\mu(n) := \begin{cases} 1 & \text{for } n = 1 \\ 0 & \text{if some } e_i > 1 \\ (-1)^r & \text{if } e_1 = e_2 = \ldots = e_r = 1 \end{cases}$$

Example: $\mu(2) = \mu(3) = -1$, $\mu(4) = 0$, $\mu(5) = -1$

$\mu(6) = 1$

**Lemma:** For $n \in \mathbb{Z}_{\geq 1}$ we have

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}$$

Proof: First, we check that the lemma is correct for $n=1$:

$$\sum_{d \mid 1} \mu(d) = \mu(1) = 1.$$

Now suppose that $n \in \mathbb{Z}_{>1}$ has the prime factorization

$$n = p_1^{e_1} \cdots p_r^{e_r}, \quad e_1, \dots, e_r \geq 1.$$

Set $n^* := p_1 \cdots p_r$, the square free part of $n$.

If $d \mid n$ and $d \nmid n^*$ then $d$ has a prime divisor of multiplicity $> 1$. Then $\mu(d) = 0$. Hence,

$$\sum_{d \mid n} \mu(d) = \sum_{d \mid n^*} \mu(d).$$

Now we can easily compute:

$$\sum_{d \mid n^*} \mu(d) = \sum_{d \mid p_1 \cdots p_r} \mu(d) = \sum_{I \subset \{1,\dots,r\}} \mu\Big(\prod_{i \in I} p_i\Big)$$

sum over all subsets of $\{1,\dots,r\}$

$$= \sum_{I \subset \{1,\dots,r\}} (-1)^{|I|} = \sum_{k=0}^{r} (-1)^k \cdot \binom{r}{k} = (1-1)^r = 0$$

This finishes the proof of the lemma $\blacksquare$

**Theorem:** (Möbius inversion formula)

Let $f, F : \mathbb{Z}_{\geq 1} \to \mathbb{C}$ be such that

$$(\text{✿}) \quad F(n) = \sum_{d \mid n} f(d), \qquad n \in \mathbb{Z}_{\geq 1}.$$

Then

$$(\text{✿✿}) \quad f(n) = \sum_{d \mid n} \mu(d) F(n/d).$$

Moreover, (✿✿) implies (✿).

Proof: Let $d$ and $n$ be numbers in $\mathbb{Z}_{\geq 1}$ such that $d$ divides $n$.

Then $F(n/d) = \sum_{d' | (n/d)} f(d')$.

Therefore

$$\sum_{d|n} \mu(d) F(n/d) = \sum_{d|n} \mu(d) \sum_{d'|(n/d)} f(d')$$

Consider the set $S_n$ of all pairs $(d, d') \in \mathbb{Z}_{\geq 1}$ such that

$d | n$ and $d' | (n/d)$.

If $n$ and its divisor $d'$ are fixed, then $d$ runs over all divisors of $n/d'$.

We can change the order of summation:

$$\sum_{d|n} \sum_{d'|(n/d)} \mu(d) f(d') = \sum_{(d,d') \in S_n} \mu(d) f(d') = \sum_{d'|n} \sum_{d|(n/d')} f(d') \mu(d)$$

$$\sum_{d' \mid n} \sum_{d \mid \frac{n}{d'}} f(d')\,\mu(d) =$$

$$= \sum_{d' \mid n} f(d') \left( \sum_{d \mid \frac{n}{d'}} \mu(d) \right) =$$

Recall:

By the Lemma

$$\sum_{d \mid \frac{n}{d'}} \mu(d) = \begin{cases} 1, & \frac{n}{d'} = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= f(n).$$

This finishes the proof of the inversion formula.

Now we will prove the second statement of the theorem

Let $F: \mathbb{Z}_{\geq 1} \to \mathbb{C}$ be any function.

Set $f(n) := \sum_{d \mid n} \mu(d) F\left(\frac{n}{d}\right)$.

Then for $n \in \mathbb{Z}_{\geq 1}$

$$\sum_{d \mid n} f(d) = \sum_{d \mid n} \sum_{d' \mid d} \mu(d') F\left(\frac{d}{d'}\right) =$$

We make the change of variables in summation

$$\{(d, d') \mid d \mid n,\ d' \mid d\} \xrightarrow{\sim} \{(d'', d') \mid d'' \mid n,\ d' \mid \tfrac{n}{d''}\}$$

$$(d, d') \mapsto \left(\frac{d}{d'}, d'\right)$$

The inverse map is given by $(d'', d') \mapsto (d' \cdot d'', d')$

$$= \sum_{d'' \mid n} \sum_{d' \mid \frac{n}{d''}} \mu(d') \cdot F(d'') = \sum_{d'' \mid n} F(d'') \sum_{d' \mid (n/d'')} \mu(d') = F(n)$$

Use lemma

This finishes the proof of the Theorem $\blacksquare$