Discrete mathematics MATH-260

Inclusion-Exclusion formula

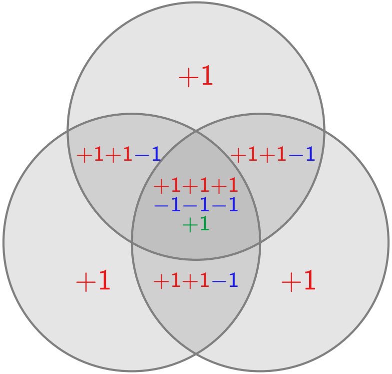
Theorem (Inclusion-Exclusion formula)

Let $A_1, \ldots, A_n$ be finite sets. Then, the following holds

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{1\leq i\leq n} |A_i| - \sum_{1\leq i<j\leq n} |A_i \cap A_j| + \sum_{1\leq i<j<k\leq n} |A_i \cap A_j \cap A_k| - \ldots$$
$$+ (-1)^{n-1}|A_1 \cap A_2 \cap \cdots \cap A_n|.$$

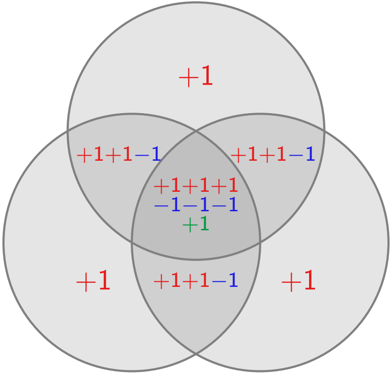
Example $n = 3$

$$|A_1 \cup A_2 \cup A_3| = + \sum_{1 \le i \le 3} |A_i| - \sum_{1 \le i < j \le 3} |A_i \cap A_j| + \sum_{1 \le i < j < k \le 3} |A_i \cap A_j \cap A_k|$$



Example $n = 3$

$$|A_1 \cup A_2 \cup A_3| = + \sum_{1 \le i \le 3} |A_i| - \sum_{1 \le i < j \le 3} |A_i \cap A_j| + \sum_{1 \le i < j < k \le 3} |A_i \cap A_j \cap A_k|$$



Each element of $A_1 \cup A_2 \cup A_3$ is counted exactly once.

Let $B_1, \dots B_m$ be finite sets and $w_1, \dots w_m$ be real numbers. Then

$$\sum_{i=1}^{m} w_i \, |B_i| = \sum_{i=1}^{m} \sum_{b \in B_i} w_i = \sum_{b \in B} \sum_{\substack{\text{indices } i \\ \text{such that } b \in B_i}} w_i,$$

where $B = \bigcup_{i=1}^{m} B_i$.

Proof of the inclusion-exclusion formula

Suppose that an element $a \in \bigcup_{i=1}^{n} A_i$ belongs to exactly k different sets.
How many times did we count a in the inclusion-exclusion formula

$$\sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \ldots \quad ?$$

Proof of the inclusion-exclusion formula

Suppose that an element $a \in \bigcup_{i=1}^{n} A_i$ belongs to exactly k different sets.
How many times did we count a in the inclusion-exclusion formula

$$\sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \dots \quad ?$$

1-st sum	2-nd sum	...	ℓ-th sum	...
$+\binom{k}{1}$	$-\binom{k}{2}$	...	$(-1)^{\ell-1}\binom{k}{\ell}$	...

Proof of the inclusion-exclusion formula

Suppose that an element $a \in \bigcup_{i=1}^{n} A_i$ belongs to exactly k different sets.
How many times did we count a in the inclusion-exclusion formula

$$\sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \dots \quad ?$$

1-st sum	2-nd sum	...	ℓ-th sum	...
$+\binom{k}{1}$	$-\binom{k}{2}$	...	$(-1)^{\ell-1}\binom{k}{\ell}$	...

We have

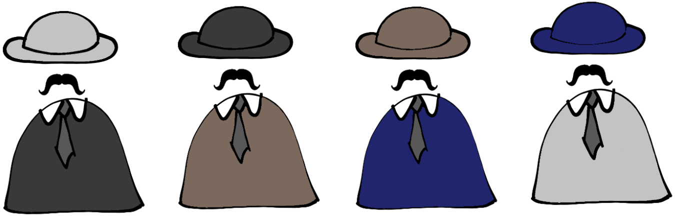
$$\sum_{\ell=1}^{n} (-1)^{\ell-1} \binom{k}{\ell} = 1.$$

Therefore, each element a is counted exactly once. This finishes the proof.

Applications of the formula

1. The hat-check girl problem

A hat-check girl completely loses track of which of n hats belong to which owners, and hands them back at random to their n owners as the latter leave. What is the probability p_n that nobody receives their own hat back?



Number of permutations without fixed points

Different formulation of the question:

Find the number of permutations of the set $[n]$ without fixed points.

Solution

Let A be the set of all permutations and A_i be the set of permutations of the set $[n]$ for which i is a fixed point. The number of permutations with no fixed points is

$$|A| - \left| \bigcup_{i=1}^{n} A_i \right| .$$

We apply the inclusion-exclusion principle to compute the cardinality of $|\bigcup_{i=1}^{n} A_i|$.

Solution

Cardinalities of intersections

A is the set of all permutations of $[n]$.
A_i is the set of permutations for which i is a fixed point.
$A_i \cap A_j$ is the set of permutations for which i and j are fixed points.
And so on

Solution

Cardinalities of intersections

A is the set of all permutations of $[n]$.
A_i is the set of permutations for which i is a fixed point.
$A_i \cap A_j$ is the set of permutations for which i and j are fixed points.
And so on
One can see that:

$$
\begin{aligned}
|A| &= n! \\
|A_i| &= (n-1)! \\
|A_i \cap A_j| &= (n-2)! \\
|A_i \cap A_j \cap A_k| &= (n-3)! \\
&\dots
\end{aligned}
$$

Solution

Altogether, this gives

$$\begin{aligned}
|A| - \left|\bigcup_{i=1}^{n} A_i\right| &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \dots \\
&= n! - \frac{n!(n-1)!}{1!(n-1)!} + \frac{n!(n-2)!}{2!(n-2)!} - \dots \\
&= n!(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^n \frac{1}{n!}) \\
&\approx n! \exp(-1).
\end{aligned}$$

Thus we see that the probability p_n that nobody receives their own hat back is

$$p_n = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^n \frac{1}{n!}$$

As n goes to infinity this number converges to $\frac{1}{e} \approx 0.37$.

Applications of the formula. Euler's totient function

Definition

In number theory, Euler's totient function $\phi(n)$ counts the positive integers up to a given integer n that are relatively prime to n.

Example 1

Among the numbers $\{1, 2, 3, 4, 5, 6\}$ only 1 and 5 are coprime to 6. Therefore, we find that $\phi(6) = 2$.

Example 2

If p is a prime number then $\phi(p) = p - 1$ and $\phi(p^k) = p^k - p^{k-1}$.

Formula for Euler's totient function

Proposition

Suppose that a number n has the prime factorization $n = p_1^{k_1} \cdots p_m^{k_m}$. Then

$$\phi(n) = n \prod_{i=1}^{m} (1 - \frac{1}{p_i}).$$

Proof

Let A be the set of all numbers in $[n]$ *not coprime* with n.
Let A_i be the set of all numbers in $[n]$ divisible by p_i.
Then $A = \bigcup_{i=1}^{m} A_i$ and $|A_i| = \frac{n}{p_i}$, $|A_i \cap A_j| = \frac{n}{p_i p_j}$, and so on.

By the inclusion-exclusion formula we find

$$\begin{aligned}\phi(n) =& n - |A| \\ =& n - \sum_{1 \leq i \leq m} \frac{n}{p_i} + \sum_{1 \leq i < j \leq m} \frac{n}{p_i p_j} - \sum_{1 \leq i < j < k \leq m} \frac{n}{p_i p_j p_k} + \ldots = n \prod_{i=1}^{m} (1 - \frac{1}{p_i}).\end{aligned}$$