Discrete mathematics MATH-260

Binomial coefficients

first meeting

Definition

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Notation

Let A be a finite set and k be a nonnegative integer. Then $\binom{A}{k}$ is the set of k-element subsets of A. We have $\left|\binom{A}{k}\right| = \binom{|A|}{k}$.

Pascal's triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & & & 1 & & & \\
 & & 1 & & & & & & 1 & & \\
 & 1 & & & & & & & & 1 & \\
1 & & & & & & & & & & 1
\end{array}
$$

Pascal's triangle

					1					
				1		1				
			1		2		1			
		1						1		
	1								1	
1										1

Pascal's triangle

1

1 1

1 2 1

1 3 3 1

1 1

1 1

Pascal's triangle

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1										1

Pascal's triangle

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Pascal's triangle

Row						
0	$\binom{0}{0} = 1$					
1	$\binom{1}{0} = 1$	$\binom{1}{1} = 1$				
2	$\binom{2}{0} = 1$	$\binom{2}{1} = 2$	$\binom{2}{2} = 1$			
3	$\binom{3}{0} = 1$	$\binom{3}{1} = 3$	$\binom{3}{2} = 3$	$\binom{3}{3} = 1$		
4	$\binom{4}{0} = 1$	$\binom{4}{1} = 4$	$\binom{4}{2} = 6$	$\binom{4}{3} = 4$	$\binom{4}{4} = 1$	
5	$\binom{5}{0} = 1$	$\binom{5}{1} = 5$	$\binom{5}{2} = 10$	$\binom{5}{3} = 10$	$\binom{5}{4} = 5$	$\binom{5}{5} = 1$

Pascal's triangle

Proposition

The following identities hold:

1. $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.
2. $\binom{n}{k}$ is the k-th element in the n-th line of Pascal's triangle.

Pascal's triangle

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Idea of the proof

Each subset of $[n+1]$ either contains element $n+1$ of not.
Number of $(k+1)$-element subsets containing $n+1$ is $\binom{n}{k}$.
Number of $(k+1)$-element subsets not containing $n+1$ is $\binom{n}{k+1}$.

Proposition

The number of subsets of an n-element set is 2^n, since we have

$$2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}.$$

The number of subsets of an n-element set having odd cardinality is 2^{n-1}. The number of subsets of an n-element set having even cardinality is 2^{n-1}.

Proof

Equal number of subsets of odd and even cardinality

Consider the map $\phi : 2^{[n]} \to 2^{[n]}$ defined by

$$\phi(A) = A \Delta \{1\} = \begin{cases} A \setminus \{1\}, & \text{if } 1 \in A \\ A \cup \{1\}, & \text{otherwise.} \end{cases}$$

Cardinalities of subsets A and $\phi(A)$ always have different parity.
Since $\phi \circ \phi = \mathrm{id}$ we deduce that ϕ is a bijection between the set of "odd" subsets and the set of "even" subsets. $\square$

Binomial theorem

Theorem

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \ldots + \binom{n}{n}x^n = \sum_{i=0}^{n} \binom{n}{i} x^i.$$

Proposition

Assume we have k identical objects and n different persons. Then, the number of ways in which one can distribute this k objects among the n persons equals

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}.$$

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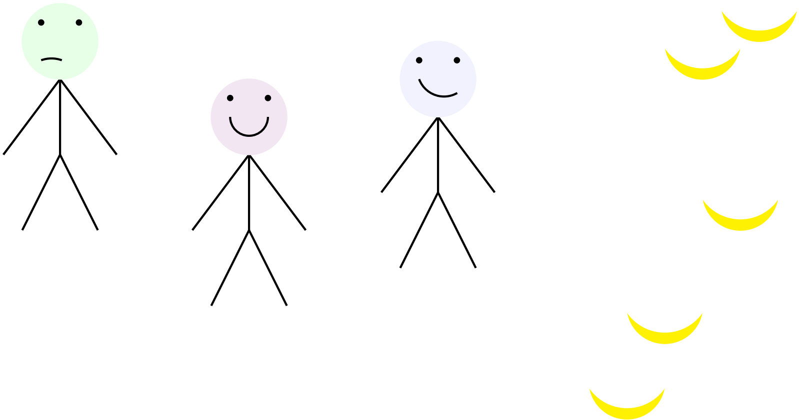
Remark

Equivalently, it is a number of solutions of the equation $x_1 + \ldots + x_n = k$ in nonnegative integers.

Distribute 5 identical objects among 3 different persons



Proof

Let $\mathcal{A}$ be the set of all solutions of the equation $x_1 + \ldots + x_n = k, x_i \in \mathbb{Z}_{\geq 0}$.
Let $\mathcal{B}$ be the set of all subsets of cardinality $n-1$ in $[k+n-1]$.
We construct a bijection $\psi : \mathcal{A} \to \mathcal{B}$ in the following way:

$$A := (x_1, \ldots, x_n) \mapsto B := \{x_1 + 1, x_1 + x_2 + 2, \ldots, x_1 + x_2 + \ldots + x_{n-1} + n - 1\}.$$

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Map ψ explained by a picture

We have $x_1 + x_2 + \ldots x_n = k$. Consider the following string

$$\underbrace{\bullet \cdots \bullet}_{x_1 \text{ bullets}} + \underbrace{\bullet \cdots \bullet}_{x_2 \text{ bullets}} + \ldots + \underbrace{\bullet \cdots \bullet}_{x_n \text{ bullets}}$$

This string consists of $k + n - 1$ symbols: k "$\bullet$" and $n-1$ "$+$".

$$\text{Set } B = \{ \text{ positions of "}+\text{" in the string}\}.$$

Proof

We show that ψ is a bijection by computing its inverse map. Let B be an element of $\mathcal{B}$. Suppose that

$$1 \leq b_1 < b_2 < \cdots < b_{n-1} \leq k + n - 1$$

are the elements of B written in the increasing order. Then the preimage $\psi^{-1}(B)$ is an n-tuple of integers $(x_1, \ldots, x_n)$ defined by

$$\begin{aligned} x_1 &= b_1 - 1 \\ x_i &= b_i - b_{i-1} - 1, \quad i = 2, \ldots, n-1 \\ x_n &= k + n - 1 - b_{n-1}. \end{aligned}$$

It is easy to see from these equations that the numbers $x_i, i = 1, \ldots n$, are non-negative integers and $x_1 + \ldots + x_n = k$.
Since there is a bijection between sets $\mathcal{A}$ and $\mathcal{B}$, their cardinalities are equal and

$$|\mathcal{A}| = |\mathcal{B}| = \binom{k+n-1}{n-1}.$$