Discrete mathematics MATH-260

Basics of counting

formalized by set theory

Finite sets

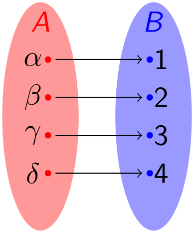
Notation

Let A be a finite set. We denote by $|A|$ the *cardinality* of A, i. e. the number of elements in the set.

Definition

Denote by $[n]$ the set of first n natural numbers: $[n] := \{1, 2, \ldots, n\}$.

Bijections (or 1 : 1 correspondences)



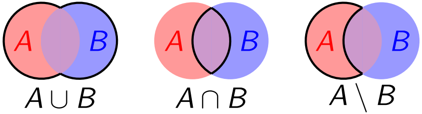
Theorem

If there exists a bijection between finite sets A and B then $|A| = |B|$.

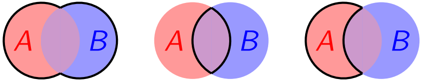
Operations with finite sets

- union
- intersection
- product
- exponentiation
- quotient

Union, intersection, set difference



Union, intersection, set difference

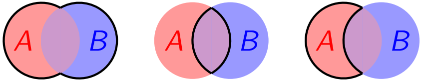


$A \cup B$ $A \cap B$ $A \setminus B$

Exercise

Suppose that you know $|A|$ and $|B|$. What can you say about $|A \cup B|$, $|A \cap B|$, and $|A \setminus B|$?

Union, intersection, set difference



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Exercise

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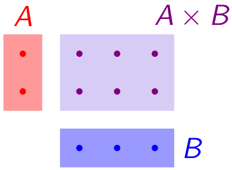
Addition rule

If $A \cap B = \emptyset$ then $|A \cup B| = |A| + |B|$.

Cartesian product

Definition

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}$$

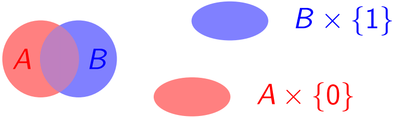


Product rule

$$|A \times B| = |A| \cdot |B|.$$

Disjoint union

$$A \sqcup B = A \times \{0\} \cup B \times \{1\}$$



Theorem

$$|A \sqcup B| = |A| + |B|.$$

Exponential object

Definition

$$A^B := \{f \mid f \text{ is a function from } B \text{ to } A\}$$

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$$|A^B| = |A|^{|B|}$$

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Exercise

What is the number of n-letter words in an m-letter alphabet?

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What is the number of n-letter words in an m-letter alphabet?

Exercise

The set of subsets of a set A is often denoted by 2^A. Can you guess why?

Example: number of permutations

Definition

A permutation π of a set A is a bijection $\pi : A \to A$

Notation

The set of permutations of $[n]$ is denoted by S_n.

Theorem

$$|S_n| = n!$$

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Example: number of permutations

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A permutation π of a set A is a bijection $\pi : A \to A$

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Exercise
Construct a $1:1$ map between S_n and the product $[1] \times [2] \times \cdots \times [n]$.

Counting techniques

- Combinatorial method (seen today)
- Induction
- Generating series
- Linear algebra method
- Probabilistic method