# Bonferroni's inequalities

**Theorem:** Let $A_1, \dots, A_n$ be finite sets.

Let $k \in \{1, \dots, n\}$ be an odd integer.

Then

$$\left| \bigcup_{i=1}^{n} A_i \right| \leq \sum_{s=1}^{k} \sum_{\substack{J \subset \{1,\dots,n\}: \\ |J| = s}} (-1)^{s-1} \cdot \left| \bigcap_{j \in J} A_j \right| .$$

Let $\ell \in \{1, \dots, n\}$ be an even integer.

Then

$$\left| \bigcup_{i=1}^{n} A_i \right| \geq \sum_{s=1}^{\ell} \sum_{\substack{J \subset \{1,\dots,n\}: \\ |J| = s}} (-1)^{s-1} \cdot \left| \bigcap_{j \in J} A_j \right| .$$

Proof: By induction on $n$

Basis of induction:

$n=1$: $|A_1| = |A_1|$

$n=2$: $|A_1|+|A_2|-|A_1 \cap A_2| = |A_1 \cup A_2| \leq |A_1|+|A_2|$

Step of induction:

Suppose that Bonferroni's inequalities are correct for $n-1$ sets

Let $k$ be an odd / even integer in $\{1,\dots,n-1\}$

$$\sum_{s=1}^{k}\ \sum_{\substack{J\subset\{1,\dots,n\}:\\ |J|=s}} (-1)^{s-1}\cdot\left|\bigcap_{j\in J}A_j\right| =$$

$$\sum_{s=1}^{k}\left(\ \sum_{\substack{J\subset\{1,\dots,n-1\}:\\ |J|=s}} (-1)^{s-1}\cdot\left|\bigcap_{j\in J}A_j\right| + \sum_{\substack{J\subset\{1,\dots,n-1\}:\\ |J|=s-1}} (-1)^{s-1}\cdot\left|\bigcap_{j\in J}(A_j\cap A_n)\right|\right)$$

$$\sum_{s=1}^{k}\ \sum_{\substack{J\subset\{1,\dots,n-1\}:\\ |J|=s}} (-1)^{s-1}\cdot\left|\bigcap_{j\in J}A_j\right| + |A_n| - \sum_{s=1}^{k-1}\ \sum_{\substack{J\subset\{1,\dots,n-1\}:\\ |J|=s}} (-1)^{s}\cdot\left|\bigcap_{j\in J}(A_j\cap A_n)\right|.$$

By the induction hypothesis:

$$\sum_{s=1}^{k} \sum_{\substack{J \subset \{1,\dots,n-1\}: \\ |J| = s}} (-1)^{s-1} \cdot \left| \bigcap_{j\in J} A_j \right| \leq / \geq \left| \bigcup_{j=1}^{n-1} A_j \right|$$

and

$$\sum_{s=1}^{k-1} \sum_{\substack{J \subset \{1,\dots,n-1\}: \\ |J| = s}} (-1)^{s} \cdot \left| \bigcap_{j\in J} (A_j \cap A_n) \right| \geq / \leq \left| \bigcup_{j=1}^{n-1} (A_j \cap A_n) \right|.$$

Therefore, we arrive at:

$$\sum_{s=1}^{k} \sum_{\substack{J \subset \{1,\dots,n\}: \\ |J|=s}} (-1)^{s-1} \cdot \left| \bigcap_{j\in J} A_j \right| =$$

$$\sum_{s=1}^{k} \sum_{\substack{J \subset \{1,\dots,n-1\}: \\ |J|=s}} (-1)^{s-1} \cdot \left| \bigcap_{j\in J} A_j \right| + |A_n| - \sum_{s=1}^{k-1} \sum_{\substack{J \subset \{1,\dots,n-1\}: \\ |J|=s}} (-1)^{s} \cdot \left| \bigcap_{j\in J} (A_j \cap A_n) \right|.$$

We use induction hypothesis:

$$\leq / \geq \ \left| \bigcup_{j=1}^{n-1} A_j \right| + |A_n| - \left| \left( \bigcup_{j=1}^{n-1} A_j \right) \cap A_n \right|.$$

By inclusion-exclusion formula this number equals

$\left| \left( \bigcup_{j=1}^{n-1} A_j \right) \cup A_n \right| = \left| \bigcup_{j=1}^{n} A_j \right|$. This finishes the proof $\blacksquare$